Škoda Auto University

Operational Research II

Lectures

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Literature

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Basic

- FÁBRY, J. Operational Research II for full-time and distance form of studies. Mladá Boleslav: ŠAU, 2025. 253 pp. (Draft).
- FÁBRY, J. Operational Research I for full-time and distance form of studies. Mladá Boleslav: ŠAU, 2022. 150 pp. ISBN 978-80-7654-048-4.

Recommended

- EISELT, H. and SANDBLOM, C. *Operations Research.: A Model-Based Approach*. 1st edition, Heidelberg: Springer, 2010. 446 pp. ISBN 978-3-642-10325-4.
- HILLIER, F. S. and LIEBERMAN, G. J. *Introduction to Operations Research*. 11th edition, McGraw-Hill, 2021. 964 pp. ISBN 9781260575873.
- BOUCHERIE, R. J., BRAAKSMA, A. and TIJMS, H. *Operations Research: Introduction to Models and Methods*. World Scientific, 2022. 499 pp. ISBN 9789811239342.
- RARDIN, R. L. *Optimization in Operations Research*. 2nd edition, Pearson, 2018. 1144 pp. ISBN 978-93-530-6636-9.
- Fábry, J. *Management Science*. University of Economics Prague, 2003. ISBN 80-245-0586–X (Available at https://janfabry.cz/Management-Science.pdf).

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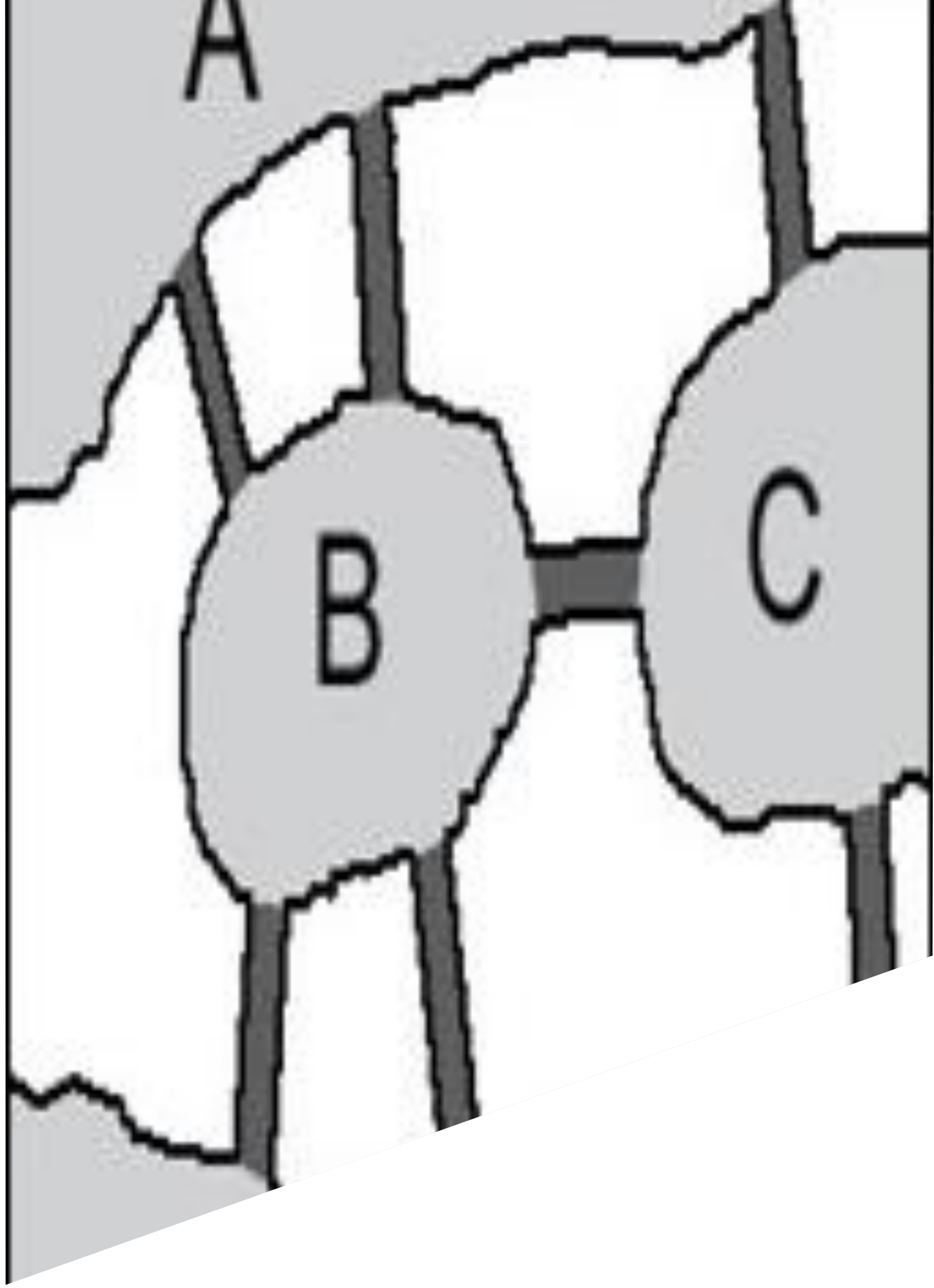
Introduction to Operational Research



Introduction to Operational Research

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Alternative Names and Related Fields

- Operational / Operations Research (OR)
- Management Science (MS)
- Operations Analysis
- Quantitative Analysis
- Quantitative Methods
- Systems Analysis
- Decision Analysis
- Decision Science
- Computer Science

Introduction to Operational Research

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Definition

1. OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problems.
2. MS/OR is the application of the scientific method to the study of the operations of large, complex organizations or activities.
3. MS/OR is the application of the scientific method to the analysis and solution of managerial decision problems.

- **Summary**
  - Application of SCIENTIFIC METHOD.
  - Study of LARGE & COMPLEX SYSTEMS.
  - Analysis of MANAGERIAL PROBLEMS.
  - Finding OPTIMAL SOLUTION.
  - Use of MATHEMATICAL MODELS.
  - Use of COMPUTERS & SPECIAL SOFTWARE.

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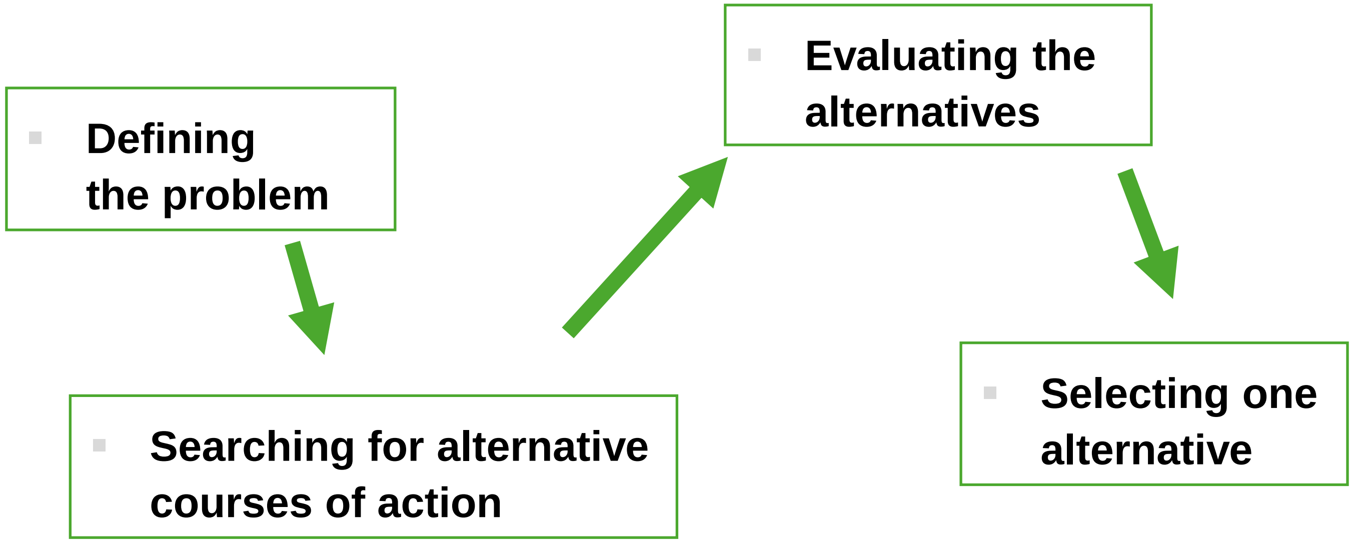
Software

- MPL for Windows
- AMPL
- Lingo (LINDO)
- XPRESS (FICO)
- CPLEX (IBM ILOG)
- AIMMS
- Gurobi
- NEOS
- MS Excel (FRONTLINE SOLVERS)
- SIMPROCESS
- SIMUL 8
- Matlab

Introduction to Operational Research

Decision making

- Two or more alternatives.
- Conclusion = Decision.
- Systematic process.



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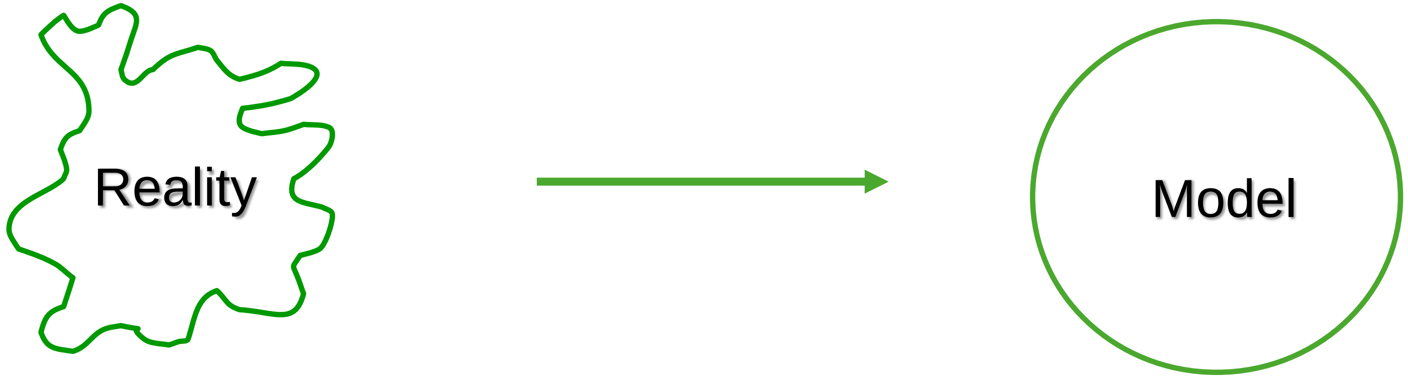
Modeling



- Finding a proper balance between the level of simplification of the model and the good representation of reality.

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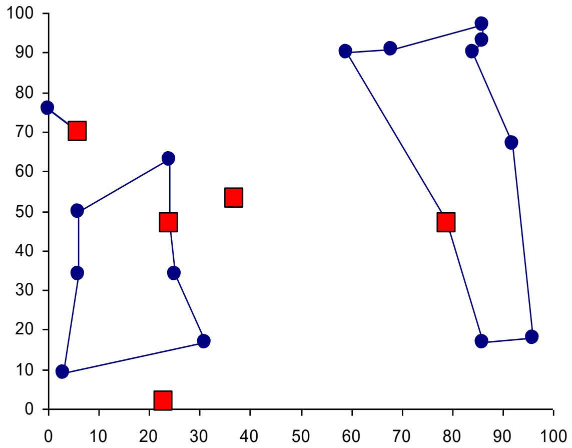
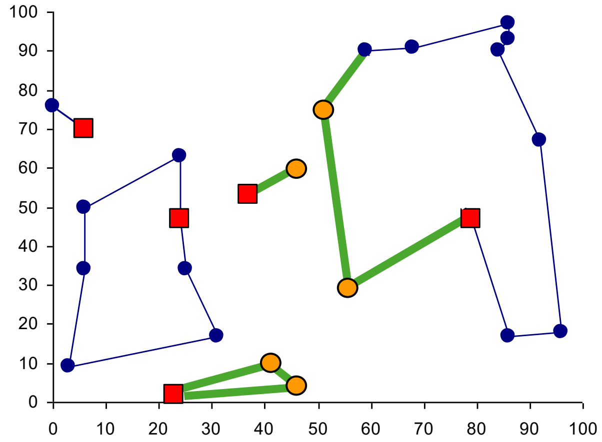
Models

- Deterministic – all parameters are known with certainty.
- Probabilistic (stochastic) – some parameters are values of random variables.



- Static – all data is known in advance (before solution process).
- Dynamic – data can be changed after solution is obtained.



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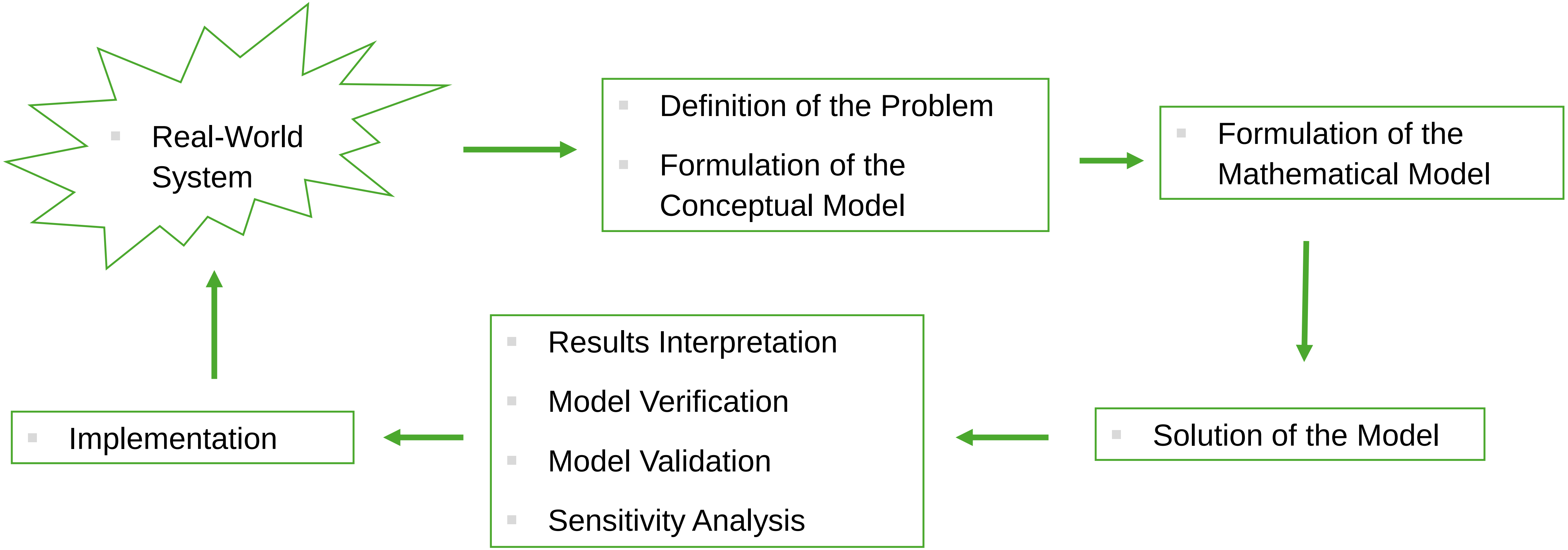
Modeling Process (Analytical Approach)



Introduction to Operational Research

Linear Programming

- **Conceptual model**
  - Processes.
  - Restrictions.
  - Objective.
- **Mathematical model**
  - Decision Variables – continuous, integral, binary.
  - Constraints – equations, inequalities.
  - Objective function – max, min.
- **Solution**
  - Feasible – satisfies all constraints.
  - Optimal – best feasible solution in terms of the objective.
  - Infeasible – does not satisfy any constraint.

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Linear Programming

- **Solution**
  - Results Interpretation – explanation of values to the others (e.g. client).
  - Model Verification – comparison of the mathematical model with the conceptual model.
  - Model Validation – comparison of results with the real expectations.
  - Sensitivity Analysis – examination of the impact of changes in inputs on outputs.
- **Implementation**
  - Use of results in real system.
- **Special situations of LP problems**
  - Unique optimal solution.
  - Multiple optimal solution.
  - No optimal solution.
  - No feasible solution.

Introduction to Operational Research

Linear Programming

- **Notation**

  $\mathbf{R}$ set of real numbers

  $\mathbf{R}_+$ set of non-negative real numbers

  $\mathbf{Z}$ sets of integers

  $\mathbf{Z}_+$ *s*ets of non-negative integers

  $\mathbf{B}$ sets of binary values $\{0,1\}$

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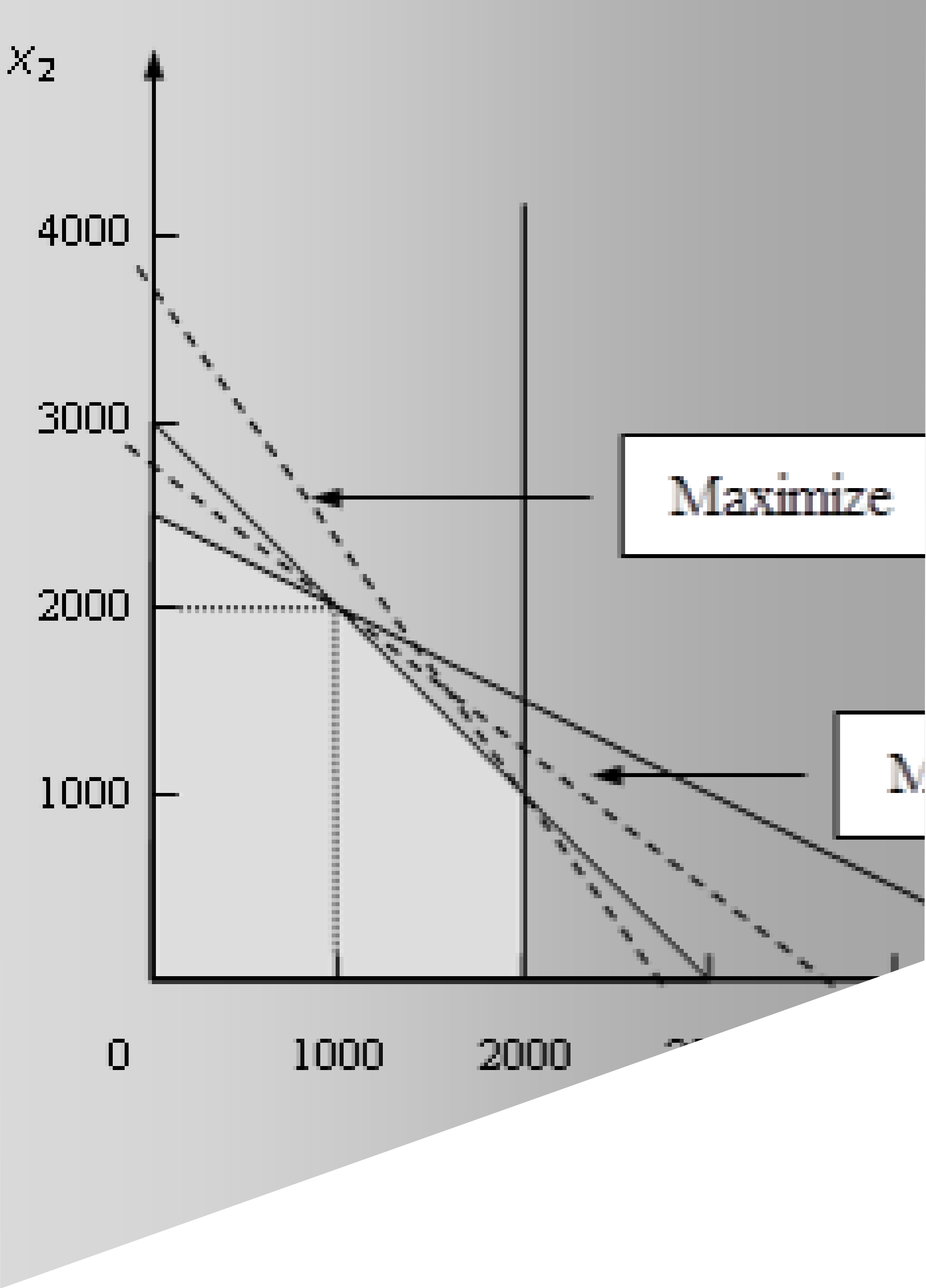
Production Planning Problems



Production Planning Problems

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Capacity Production Planning Problem

- **Example**
  - The joinery manufactures tables and chairs. The production process includes machining, grinding and assembly.
  - The table components are machined in 5 hours, ground in 4 hours and assembled in 3 hours.
  - The chair components are machined in 2 hours, ground in 3 hours and assembled in 4 hours.
  - There are 270 hours available for machining, 250 hours for grinding and 200 hours for assembly.
  - The profit per table sold is €100 and per chair sold is €60.
  - The aim is to design the production so that the total profit from the furniture sold is as high as possible (we assume that all the production is sold).

  - How does the solution change if we have to produce 4 chairs for each table produced?

Production Planning Problems

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Capacity Production Planning Problem – mathematical model formulation

- **Decision variables**

  x_1 = number of tables produced,

  x_2 = number of chairs produced.

- **Mathematical model**

$$z = 100x_1 + 60x_2 \to \max,$$

$$5x_1 + 2x_2 \le 270 \qquad \text{(machining)},$$

$$4x_1 + 3x_2 \le 250 \qquad \text{(grinding)},$$

$$3x_1 + 4x_2 \le 200 \qquad \text{(assembly)},$$

$$\left.\begin{array}{l} x_1, x_2 \ \ \ge 0 \\ x_1, x_2 - \text{ integers} \end{array}\right\} \text{or } x_1, x_2 \in \mathbb{Z}_+ .$$

Slack/surplus variables →

Equivalent set of equations

$$\begin{aligned} 5x_1 + 2x_2 + x_3 \qquad\qquad &= 270 \\ 4x_1 + 3x_2 \qquad + x_4 \qquad &= 250 \\ 3x_1 + 4x_2 \qquad\qquad + x_5 &= 200 \end{aligned}$$

$\le$ + slack variable

$\ge$ - surplus variable

Production Planning Problems

Capacity Production Planning Problem – solution

- **Optimal solution**
  - Decision variables

    $x_1 = 48$

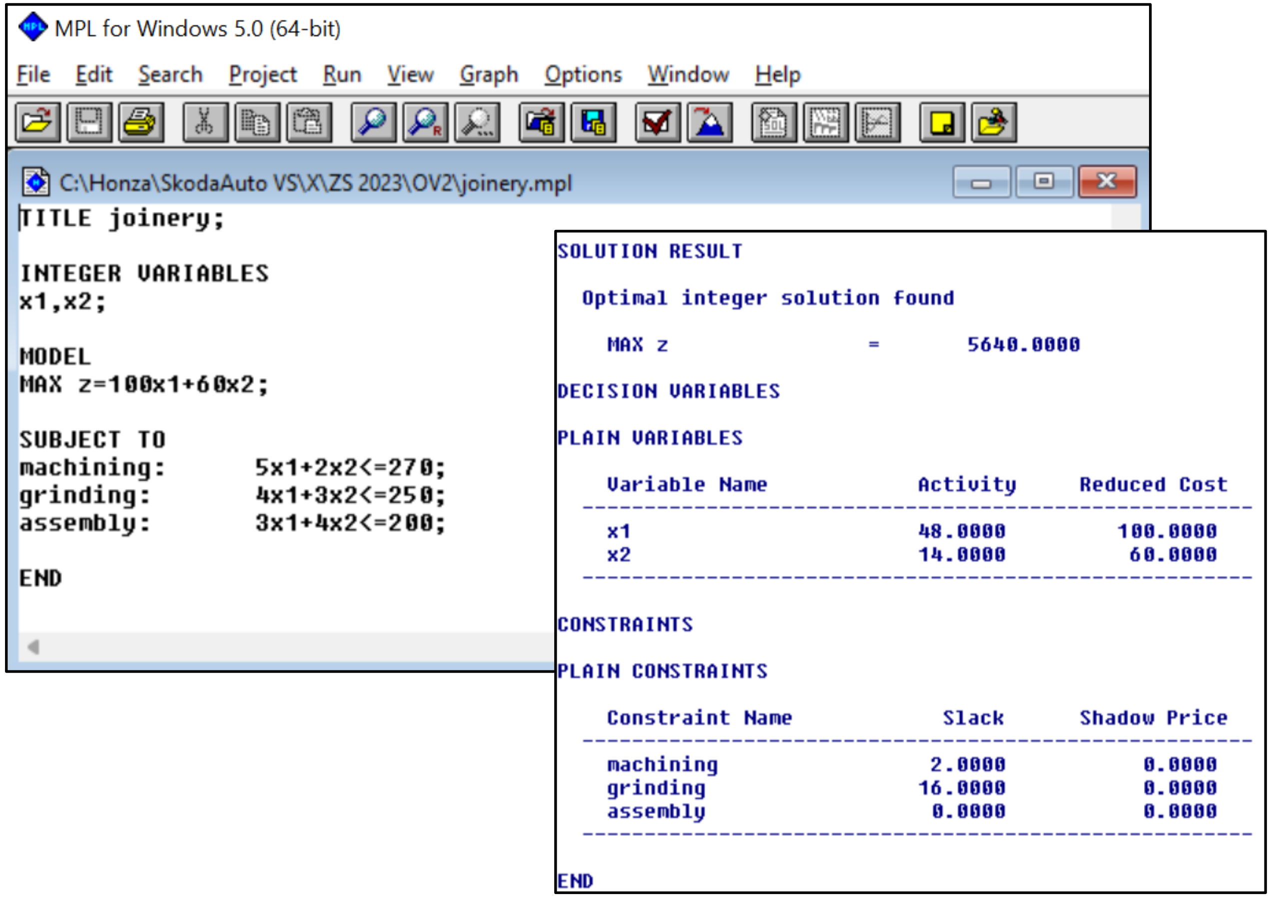
    $x_2 = 14$
  - Objective value

    $z_0 = 5640$
  - Slack/surplus variables

    $x_3 = 2$

    $x_4 = 16$

    $x_5 = 0$



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Capacity Production Planning Problem – model, solution

- **Mathematical model**
  - Additional limitation: How does the solution change if we have to produce 4 chairs for each table produced?

$$4x_1 = x_2 \qquad \text{(sets)}$$

```
TITLE joinerySets;

INTEGER VARIABLES
x1,x2;

MODEL
MAX z=100x1+60x2;

SUBJECT TO
machining:     5x1+2x2<=270;
grinding:      4x1+3x2<=250;
assembly:      3x1+4x2<=200;
productsets:   4x1-x2=0;

END
```

- **Optimal solution**
  - Decision variables

$$x_1 = 10$$
$$x_2 = 40$$

  - Objective value

$$z_0 = 3400$$

  - Slack/surplus variables

$$x_3 = 140$$
$$x_4 = 90$$
$$x_5 = 10$$

```
SOLUTION RESULT

  Optimal integer solution found

    MAX z              =        3400.0000

DECISION VARIABLES

PLAIN VARIABLES

    Variable Name             Activity       Reduced Cost
  -----------------------------------------------------
    x1                         10.0000          100.0000
    x2                         40.0000           60.0000
  -----------------------------------------------------

CONSTRAINTS

PLAIN CONSTRAINTS

    Constraint Name              Slack       Shadow Price
  -----------------------------------------------------
    machining                 140.0000            0.0000
    grinding                   90.0000            0.0000
    assembly                   10.0000            0.0000
    productsets                 0.0000            0.0000
  -----------------------------------------------------

END
```

Production Planning Problems

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Production Planning Problem with Semi-Finished Products

- **Example**
  - Firm produces products P_1, P_2 a P_3.
  - To produce 1 unit of product P_1 the firm uses 3 kg of material.
  - To produce 1 unit of product P_2 the firm uses 2 kg of material and 1 unit of product P_1.
  - To produce 1 unit of product P_3 the firm uses 2 kg of material, 2 units of product P_1 and 1 unit of product P_2.
  - There are 1000 kg of material available.
  - Products P_1 and P_2 that are used as semi-finished products can also be sold themselves.
  - Prices of goods P_1, P_2 and P_3 are 5, 10 and 30 €. The objective is to maximize total revenue from products sold.

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Production Planning Problem with Semi-Finished Products

- **Assembly table**

Entry	P_1	P_2	P_3
Material (kg)	3	2	2
P_1 (unit)	-	1	2
P_2 (unit)	-	-	1
Price (€/unit)	5	10	30

Production Planning Problems

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Production Planning Problem with Semi-Finished Products – mathematical model formulation

- **Decision variables**

  $x_i =$ number of products produced P_i $(i = 1, 2, 3)$

- **Mathematical model**

  $z = 5x_1 + 5x_2 + 10x_3 \to \max$ (total revenue),

  $3x_1 + 2x_2 + 2x_3 \leq 1000$ (raw material consumption),

  $x_1 \geq x_2 + 2x_3$ (product P_1 consumption),

  $x_2 \geq x_3$ (product P_2 consumption),

  $x_i \in \mathbf{Z}_+ \quad i = 1, 2, 3.$

```
TITLE SemiFinished;

INTEGER VARIABLES
x1,x2,x3;

MODEL
MAX z=5x1+5x2+10x3;

SUBJECT TO
Material:       3x1+2x2+2x3<=1000;
P1:             x1-x2-2x3>=0;
P2:             x2-x3>=0;

END
```

Production Planning Problems

Production Planning Problem with Semi-Finished Products – solution

- **Optimal solution**
  - Decision variables

    $x_1 = 231$ (number of products produced P_1)

    $x_2 = 77$ (number of products produced P_2)

    $x_3 = 76$ (number of products produced P_3)

  - Objective value

    $z_0 = 2300$ (maximum total revenue in €)

  - Slack/surplus variables

    $x_4 = 1$ (1 kg of raw material left)

    $x_5 = 2$ (2 units of P_1 to be sold separately)

    $x_6 = 1$ (2 units of P_2 to be sold separately)

```
SOLUTION RESULT

  Optimal integer solution found

    MAX z              =        2300.0000

DECISION VARIABLES

PLAIN VARIABLES

    Variable Name             Activity      Reduced Cost
  ---------------------------------------------------------
    x1                        231.0000            5.0000
    x2                         77.0000            5.0000
    x3                         76.0000           10.0000
  ---------------------------------------------------------

CONSTRAINTS

PLAIN CONSTRAINTS

    Constraint Name              Slack      Shadow Price
  ---------------------------------------------------------
    Material                    1.0000            0.0000
    P1                         -2.0000            0.0000
    P2                         -1.0000            0.0000
  ---------------------------------------------------------

END
```

Production Planning Problems

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Using Logical Variables in a Mathematical Model

- **Binary decision variables**
  - take the value 1 or 0,
  - they can be used in a mathematical model as logical or decision variables:
    - produced / not produced,
    - used/not used,
    - one option / another option,
    - etc.

Production Planning Problems

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Fixed-Cost Production Planning Problem

- **Example**
  - Possible production of n products on n production lines (each product on exactly one line - PL).
  - If the j-th product is produced, then a maximum of z_j pieces can be produced.
  - Fixed cost f_j has to be considered if PL_j is used (i.e. product i is produced).
  - Unit profit c_j is given for product j.
  - Standard production planning (capacity) constraints are defined.
  - The objective is to maximize total profit decreased by fixed cost.

Production Planning Problems

Fixed-Cost Production Planning Problem

- **Decision variables**

$$x_j = \begin{cases} 1 & \text{if product } j \text{ is produced } \left(\text{on } PL_j\right) \\ 0 & \text{otherwise} \end{cases}$$

$$y_j = \text{number of product } j \text{ being produced}$$

- **Objective**

$$z = \sum_{j=1}^{n} c_j y_j - \sum_{j=1}^{n} f_j x_j \quad \rightarrow \max$$

- **Constraints**

$$\sum_{j=1}^{n} a_{lj} y_j \le b_l \qquad l = 1, 2, \ldots, m \qquad \left(\text{capacity constraints}\right),$$

$$y_j \le z_j x_j \qquad j = 1, 2, \ldots, n$$

If is $y_j > 0$, then have to be $x_j = 1$
If is $x_j = 0$, then have to be $y_j = 0$

(If no z_j limit is specified, the high constant M is used instead)

$$x_j \in \mathbf{B} \qquad j = 1, 2, \ldots, n,$$

$$y_j \in \mathbf{R}_+ \qquad j = 1, 2, \ldots, n.$$

Production Planning Problems

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Production Planning with Alternative Production Sequences

- The problem is solved with the condition of validity of different sets of constraints (either-or constraints)
- **Example**
  - Three products can be produced on a machine either in the sequence $P_1 \rightarrow P_2 \rightarrow P_3$ or $P_3 \rightarrow P_2 \rightarrow P_1$.
  - Assume production of P_i takes t_i.
  - Formulate the constraints for feasible production.

Production Planning Problems

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Production Planning with Alternative Production Sequences

- **Decision variables**

$y_i =$ starting production time of product P_i

$$x = \begin{cases} 1 & \text{if sequence } P_1 \to P_2 \to P_3 \text{ is used} \\ 0 & \text{if sequence } P_3 \to P_2 \to P_1 \text{ is used} \end{cases}$$

- **Model**

$$y_1 + t_1 \le y_2 + M(1-x),$$
$$y_2 + t_2 \le y_3 + M(1-x),$$
$$y_3 + t_3 \le y_2 + Mx,$$
$$y_2 + t_2 \le y_1 + Mx,$$
$$y_i \in \mathbf{R}_+ \qquad i = 1, 2, 3,$$
$$x \in \mathbf{B}.$$

$M =$ high constant (∞)

If the first sequence is to be produced, then $x = 1$ and the first two conditions guarantee that the time sequences of the production of each product are met. The other two constraints are always satisfied (due to the high constant on the right-hand side). If the second sequence is to be produced, the explanation is similar.

Production Planning Problems

Production Planning with Range of Production Level

- **Example**
  - A company is considering whether to produce a new product or not.
  - If so, the level of production should be at least 500 units but not more than 1000 units.
- **Decision variables**

$$y = \text{production level}$$

$$x = \begin{cases} 1 & \text{if the decision to produce is yes} \\ 0 & \text{otherwise} \end{cases}$$

- **Model**

$$500x \leq y \leq 1000x,$$

$$y \in \mathbf{Z}_+,$$

$$x \in \mathbf{B}.$$

Production Planning Problems

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Planning Production on Discrete Levels

- **Example**
  - A company decides to produce either 500 or 1000 or 2000 units of certain product.
- **Decision variables**

$$y = \text{level of production}$$

$$x_i = \begin{cases} 1 & \text{if the production is set on } i\text{-th level} \\ 0 & \text{otherwise} \end{cases} \qquad i = 1, 2, 3$$

- **Model**

$$y = 500x_i + 1000x_2 + 2000x_3,$$

$$x_1 + x_2 + x_3 = 1,$$

$$y \in \mathbf{Z}_+,$$

$$x_i \in \mathbf{B} \qquad i = 1, 2, 3.$$

Production Planning Problems

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Cutting Stock Problem

- **Input – raw product (dimension)**
  - pipes, tubes (1D).
  - roles of paper or textile (1D or 2D),
  - wooden sticks or laths (1D),
  - wooden boards (1D or 2D),
  - steel plates (2D),
  - boxes (3D) – 3D rectangular packing problem.
- **Output – final or semi-finished products**
- **Objective**
  - minimization of total loss,
  - minimization of a number of raw products being cut,
  - maximization of number of final/assembled products,
  - maximization of revenue or profit ensuing from sold final/assembled products.

Production Planning Problems

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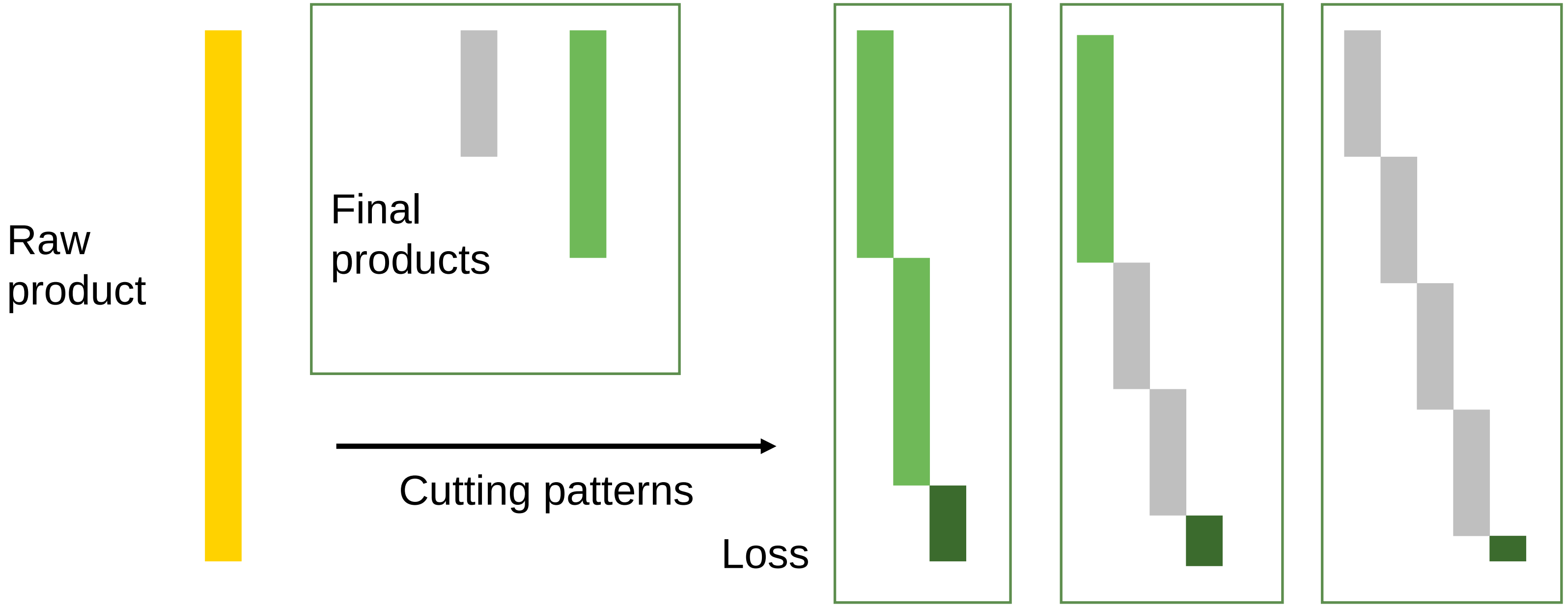
Cutting Stock Problem

- **Table of cutting patterns**
  - It contains all possibilities of cutting raw products.
  - Each cutting pattern corresponds to the variable giving a number of the raw products being cut according to this pattern.



Production Planning Problems

Cutting Stock Problem

- **Example**
  - Firm produces garden laths fence. To produce the fence for a particular order, the company needs 200 laths 140 cm long, 320 laths 80 cm long and 480 laths 60 cm long.
  - Only standard 300 cm long laths are available in the warehouse.
  - It is necessary to sytisfy the order while using the minimum number of standard laths.

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Cutting Stock Problem

- **Table of cutting patterns**

Pattern	1	2	3	4	5	6	7	8
140 cm (pcs)	2	1	1	1	0	0	0	0
80 cm (pcs)	0	2	1	0	3	2	1	0
60 cm (psc)	0	0	1	2	1	2	3	5
Loss (in cm)	20	0	20	40	0	20	40	0

Production Planning Problems

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Cutting Stock Problem – mathematical model formulation

- **Parameters**

  $m =$ number of types of shorter parts

  $n =$ number of cutting patterns

  $b_i =$ required number of parts of i-th type $\left(i = 1, 2, \ldots, m\right)$

  $a_{ij} =$ number of parts of i-th type obtained according to j-th cutting pattern $\left(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n\right)$

- **Decision variables**

  $x_j =$ number of pieces of original material cut according to j-th cutting pattern

- **Objective**

$$z = \sum_{j=1}^{n} x_j \rightarrow \min \qquad \text{(minimizing the number of cut pieces of original material)}$$

- **Constraints**

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1, 2, \ldots, m,$$

$$x_j \in \mathbf{Z}_+ \qquad j = 1, 2, \ldots, n.$$

```
TITLE CuttingStock;

OPTIONS
EXCELWORKBOOK="CuttingStock.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("part");
j:=EXCELRANGE("pattern");

DATA
a[i,j]:=EXCELRANGE("cuts");
b[i]:=EXCELRANGE("requirement");

INTEGER VARIABLES
x[j] EXPORT TO EXCELRANGE("number");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(j:x[j]);

SUBJECT TO
part[i]: sum(j:a[i,j]*x[j])=b[i];

END
```

Production Planning Problems

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Cutting Stock Problem – solution

- **Optimal solution**
  - Decision variables

$x_1 = 20$ (number of 3 m laths cut using the 1st cutting pattern)

$x_2 = 160$ (umber of 3 m laths cut using the 2nd cutting pattern)

$x_8 = 96$ (number of 3 m laths cut using the 8th cutting pattern)

$x_3, x_4, x_5, x_6, x_7 = 0$ (other cutting patterns were not used)

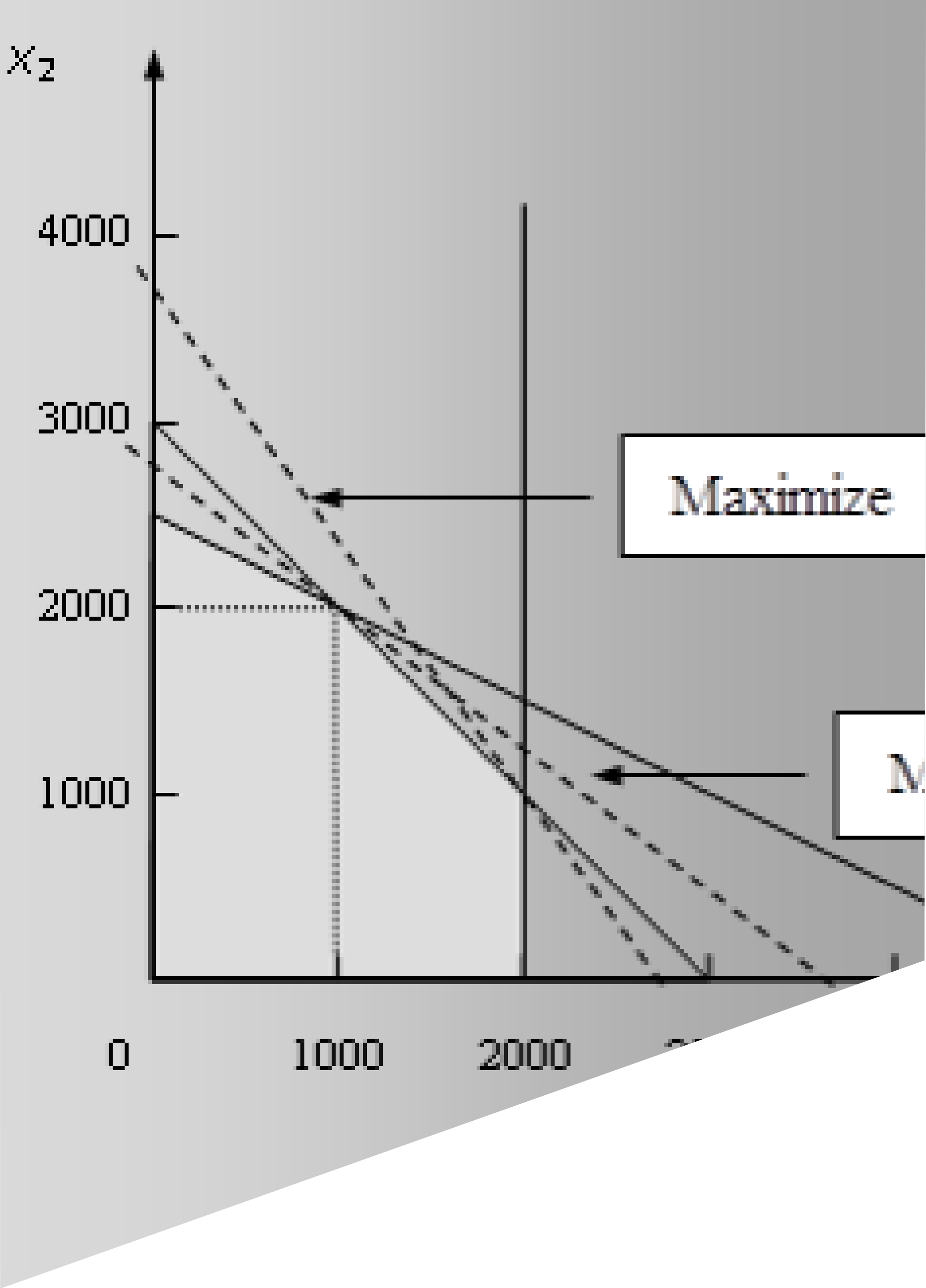
  - Objective value

$z_0 = 276$ (total number of cut 3 m laths)

```
SOLUTION RESULT

  Optimal integer solution found

    MIN z                  =        276.0000

DECISION VARIABLES

VARIABLE x[j] :

   j             Activity     Reduced Cost
  ----------------------------------------
   P1             20.0000           1.0000
   P2            160.0000           1.0000
   P8             96.0000           1.0000
  ----------------------------------------
```

3

Assignment Problem



Assignment Problem

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Linear Assignment Problem

- **Definition of the problem**
  - Two sets of items.
  - Each item from the first set is to be assigned to exactly one item from the second set.
  - Each item from the second set is to be assigned to exactly one item from the first set.
  - The assignment of each pair of items is evaluated.
  - The objective is to maximize/minimize total value of assignment.
  - Assumption: sizes of both sets are equal (balanced problem). Otherwise, data or a mathematical model must be changed (unbalanced problem)

Assignment Problem

Linear Assignment Problem

- **Example**
  - Relay race for 5-member teams is organized.
  - A member of each team will be competing in one discipline. You are going to build a strongest team. In the table, the seasonal best performances (in minutes) of candidates are given.

Time (min)	Run	Swim	Bike	Inline	Ski
Mike	75	25	202	130	165
Jack	87	24	198	127	173
Peter	68	19	195	121	164
Sean	91	20	207	122	182
Paul	80	28	215	125	172
Simon	78	22	197	125	180
Tom	75	25	205	127	178
David	81	23	211	131	165

Assignment Problem

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Linear Assignment Problem – matrix minimum method

- **Feasible solution**
  - Decision variables

Time (min)	Run	Swim	Bike	Inline	Ski
Mike	75	25	202	130	165
Jack	87	24	198	127	173
Peter	68	19	195	121	164
Sean	91	20	207	122	182
Paul	80	28	215	125	172
Simon	78	22	197	125	180
Tom	75	25	205	127	178
David	81	23	211	131	165

  - Objective value

$z = 578$ (time to complete the relay)

Assignment Problem

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Linear Assignment Problem – basic mathematical model formulation

- **Parameters**

$$c_{ij} \;= \text{evaluation of the pair of } i \text{ and } j$$

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if } i \leftrightarrow j \\ 0 & \text{otherwise} \end{cases}$$

Assignment Problem

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Linear Assignment Problem – basic mathematical model formulation

- Balanced problem

$$z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \rightarrow \min,$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2, \ldots, n,$$

$$x_{ij} \in \mathbf{B} \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, n \end{matrix}.$$

- Unbalanced problem ($m > n$)

$$z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \rightarrow \min,$$

$$\sum_{j=1}^{n} x_{ij} \le 1 \qquad i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad j = 1, 2, \ldots, n,$$

$$x_{ij} \in \mathbf{B} \qquad \begin{matrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{matrix},$$

or using $(m - n)$ dummy items.

```
TITLE Relay;

OPTIONS
EXCELWORKBOOK="Relay.xlsm";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("sportsman");
j:=EXCELRANGE("discipline");

DATA
c[i,j]:=EXCELRANGE("time");

BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("relay");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
sportsman[i]:  sum(j:x[i,j])<=1;
discipline[j]: sum(i:x[i,j])=1;

END
```

Assignment Problem

Linear Assignment Problem – solution

- **Optimal solution**
  - Decision variables

Time (min)	Run	Swim	Bike	Inline	Ski
Mike	75	25	202	130	165
Jack	87	24	198	127	173
Peter	68	19	195	121	164
Sean	91	20	207	122	182
Paul	80	28	215	125	172
Simon	78	22	197	125	180
Tom	75	25	205	127	178
David	81	23	211	131	165

  - Objective value

$z_0 = 575$ (minimum time to complete the relay)

```
SOLUTION RESULT

  Optimal integer solution found

    MIN z                  =        575.0000

DECISION VARIABLES

VARIABLE x[i,j] :

   i      j                  Activity     Reduced Cost
  ---------------------------------------------------
   Jack   Bike                1.0000         198.0000
   Peter  Run                 1.0000          68.0000
   Sean   Inline              1.0000         122.0000
   Simon  Swim                1.0000          22.0000
   David  Ski                 1.0000         165.0000
  ---------------------------------------------------
```

Assignment Problem

Škoda Auto University

Bottleneck Assignment Problem

- **Definition of the problem**
  - Let n jobs and n parallel machines be given.
  - The coefficient c_{ij} is the time needed for machine j to complete job i. The objective is to minimize the latest completion time. (All machines start working on jobs at the same time).

Assignment Problem

Škoda Auto University

Bottleneck Assignment Problem

- **Example**
  - The project consists of 5 independent parts. In the company 5 departments can manage the parts individually.
  - Historical data shows average times (in days) departments finished similar tasks (see the table).
  - N.A. represents the fact the department did not work on such task in the past.
  - The company wants to finish the whole project as soon as possible.

Time (days)	Part 1	Part 2	Part 3	Part 4	Part 5
Dept 1	25	15	N.A.	17	25
Dept 2	22	N.A.	22	20	22
Dept 3	20	18	25	16	23
Dept 4	N.A.	20	30	21	28
Dept 5	27	19	27	18	N.A.

- We replace the N.A. values with high prohibitive constants, e.g. 1000.

Assignment Problem

Škoda Auto University

Bottleneck Assignment Problem – mathematical model formulation

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if job } i \text{ is assigned to department } j \\ 0 & \text{otherwise} \end{cases}$$

$$T = \text{last job completion time}$$

- **Model**

$$T \rightarrow \min,$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, 2, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2, \ldots, n,$$

$$c_{ij} x_{ij} \le T \qquad i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, n,$$

$$x_{ij} \in \mathbf{B} \qquad i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, n,$$

$$T \in \mathbf{R}_{+}$$

```
TITLE Bottleneck;

OPTIONS
EXCELWORKBOOK="Bottleneck.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("department");
j:=EXCELRANGE("part");

DATA
c[i,j]:=EXCELRANGE("time");

BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("assignment");

VARIABLES
T EXPORT TO EXCELRANGE("total");

MODEL
MIN z=T;

SUBJECT TO
bottleneck[i,j]:        c[i,j]*x[i,j]<=T;
department[i]:          sum(j:x[i,j])=1;
part[j]:                sum(i:x[i,j])=1;

END
```

Assignment Problem

Bottleneck Assignment Problem – solution

- **Optimal solution**
  - Decision variables

Time (days)	Part 1	Part 2	Part 3	Part 4	Part 5
Dept 1	25	15	1000	17	**25**
Dept 2	22	1000	**22**	20	22
Dept 3	**20**	18	25	16	23
Dept 4	1000	20	30	**21**	28
Dept 5	27	**19**	27	18	1000

  - Objective value

$z_0 = 25$ (minimum time to complete the last part)

```
SOLUTION RESULT

  Optimal integer solution found

    MIN z                    =          25.0000

DECISION VARIABLES

VARIABLE x[i,j] :

   i     j                  Activity      Reduced Cost
  ------------------------------------------------
   Dpt1  Part5                1.0000           25.0000
   Dpt2  Part3                1.0000            0.0000
   Dpt3  Part1                1.0000            0.0000
   Dpt4  Part4                1.0000            0.0000
   Dpt5  Part2                1.0000            0.0000
  ------------------------------------------------
```

Assignment Problem

Škoda Auto University

Perfect Matching Problem

- **Example**
  - Ten students go for a school trip. To assign them to double rooms, they were asked to express their preferences (see the table, 0 = min, 10 = max).
  - For $i < j$ the value p_{ij} is the preference value expressing student i wants to be in the room with student j, for $i > j$ the value p_{ij} is the preference value expressing student j wants to be in the room with student i.
  - Assign students to rooms to maximize total satisfaction of the group.

Pref	1	2	3	4	5	6	7	8	9	10
1	0	7	6	2	4	7	4	1	8	3
2	1	0	3	1	10	5	2	9	4	2
3	10	1	0	5	6	1	8	2	7	4
4	1	8	4	0	10	7	5	4	2	7
5	8	7	3	5	0	2	1	5	2	9
6	2	2	3	7	8	0	8	2	1	5
7	1	7	6	1	7	7	0	8	1	5
8	6	8	1	1	10	8	1	0	4	7
9	4	1	2	2	8	1	7	5	0	2
10	1	5	4	3	9	7	1	4	6	0

Assignment Problem

Škoda Auto University

Perfect Matching Problem – mathematical model formulation

- **Parameters**

$$p_{ij} = \begin{cases} \text{preference of student } j \text{ by student } i \ (i < j) \\ \text{preference of student } i \text{ by student } j \ (i > j) \end{cases}$$

$$c_{ij} = \text{index od satisfaction of pair of students } i \text{ a } j \ (i < j)$$

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if students } i \text{ and } j \text{ are roommates} \\ 0 & \text{otherwise} \end{cases} \qquad i < j$$

- **Model**

$$c_{ij} = p_{ij} + p_{ji} \qquad i = 1, 2, \dots, n-1; \ \ j = i+1, i+2, \dots, n,$$

$$z = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} x_{ij} \rightarrow \max,$$

$$\sum_{j<i} x_{ji} + \sum_{j>i} x_{ij} = 1 \qquad i = 1, 2, \dots, n,$$

$$x_{ij} \in \mathbf{B} \qquad i = 1, 2, \dots, n-1; \ \ j = i+1, i+2, \dots, n.$$

```
TITLE PerfectMatching;

OPTIONS
EXCELWORKBOOK="PerfectMatching.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("student");
j:=i;
pair[i,j] WHERE i<j;

DATA
p[i,j]:=EXCELRANGE("preference");
c[i,j in pair]:=p[i,j]+p[i:=j,j:=i];

BINARY VARIABLES
x[i,j in pair];

MODEL
MAX z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
student[i]: sum(j<i:x[i:=j,j:=i])+sum(j>i:x[i,j])=1;

END
```

Assignment Problem

Škoda Auto University

Perfect Matching Problem – solution

- **Optimal solution**
  - Decision variables

Pref	1	2	3	4	5	6	7	8	9	10
1	-	8	16	3	12	9	5	7	12	4
2	-	-	4	4	9	17	7	17	5	7
3	-	-	-	9	9	4	14	3	9	8
4	-	-	-	-	15	14	3	9	4	10
5	-	-	-	-	-	10	8	15	10	18
6	-	-	-	-	-	-	15	10	2	12
7	-	-	-	-	-	-	-	9	8	6
8	-	-	-	-	-	-	-	-	9	11
9	-	-	-	-	-	-	-	-	-	8
10	-	-	-	-	-	-	-	-	-	-

```
SOLUTION RESULT

  Optimal integer solution found

    MAX z                  =          75.0000

DECISION VARIABLES

VARIABLE x[i,j IN pair] :

    i    j              Activity       Reduced Cost
  -------------------------------------------------
    1    9                1.0000            12.0000
    2    8                1.0000            17.0000
    3    7                1.0000            14.0000
    4    6                1.0000            14.0000
    5   10                1.0000            18.0000
  -------------------------------------------------
```

  - Objective value

$$z_0 = 75 \qquad \text{(maximum total satisfaction)}$$

Assignment Problem

Škoda Auto University

Knapsack Problem

- **Definition of the problem**
  - Budget b is available for investments in n considered projects, where a_j is the cost for project j and c_j is its expected return.
  - The objective is to select a set of projects to maximize the total expected return while not exceeding the budget.
- **Decision variables**

$$x_j = \begin{cases} 1 & \text{if the project } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, 2, \ldots, n.$$

- **General model**

$$z = \sum_{j=1}^{n} c_j x_j \rightarrow \max,$$

$$\sum_{j=1}^{n} a_j x_j \leq b,$$

$$x_j \in \mathbf{B} \qquad j = 1, 2, \ldots, n$$

Assignment Problem

Škoda Auto University

Knapsack Problem

- **Example**
  - There are 5 projects characterized by the investment cost and return.
  - The budget 50 000 € is available to select such projects that assure the highest total return.

Projects	P1	P2	P3	P4	P5
Cost	12 000	10 000	15 000	18 000	16 000
Return	20 000	18 000	22 000	26 000	21 000

Assignment Problem

Škoda Auto University

Knapsack Problem – mathematical model formulation

- **Decision variables**

$$x_j = \begin{cases} 1 & \text{if the project } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, 2, \ldots, n.$$

- **Model**

$$z = \sum_{j=1}^{n} c_j x_j \to \max,$$

$$\sum_{j=1}^{n} a_j x_j \le b,$$

$$x_j \in \mathbf{B} \qquad j = 1, 2, \ldots, n.$$

```
TITLE Knapsack;

INDEX
j:=1..5;

DATA
a[j]:=1000 (12,10,15,18,16);
c[j]:=1000 (20,18,22,26,21);
b:=50000;

BINARY VARIABLES
x[j];

MODEL
MAX z=sum(c*x);

SUBJECT TO
cost: sum(a*x)<=b;

END
```

Assignment Problem

Škoda Auto University

Knapsack Problem – solution

- **Optimal solution**
  - Decision variables

    $x_3 = x_4 = x_5 = 1$ (we invest in the 3rd , 4th and 5th projects)

    $x_1 = x_2 = 0$ (we will not invest in projects 1 and 2)
  - Objective value

    $z_0 = 69000$ (maximum possible total return on investment)
  - Slack/surplus variables

    $x_6 = 1000$ (unused part of budget)

```
SOLUTION RESULT

  Optimal integer solution found

    MAX z                  =       69000.0000

DECISION VARIABLES

VARIABLE x[j] :

    j           Activity      Reduced Cost
  ---------------------------------------
   3             1.0000        22000.0000
   4             1.0000        26000.0000
   5             1.0000        21000.0000
  ---------------------------------------

CONSTRAINTS

PLAIN CONSTRAINTS

    Constraint Name              Slack        Shadow Price
  ------------------------------------------------------
    cost                     1000.0000            0.0000
  ------------------------------------------------------

END
```

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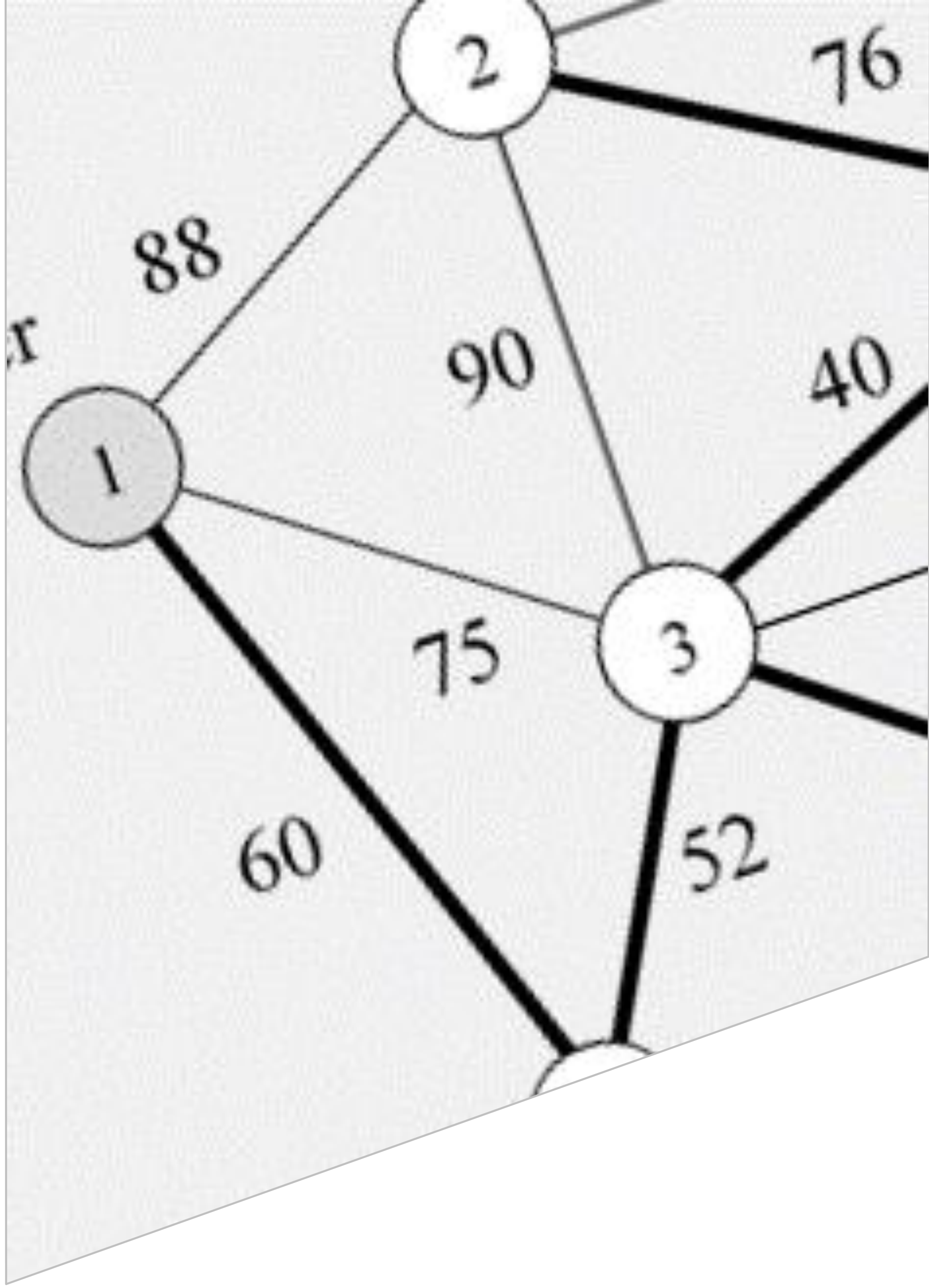
4

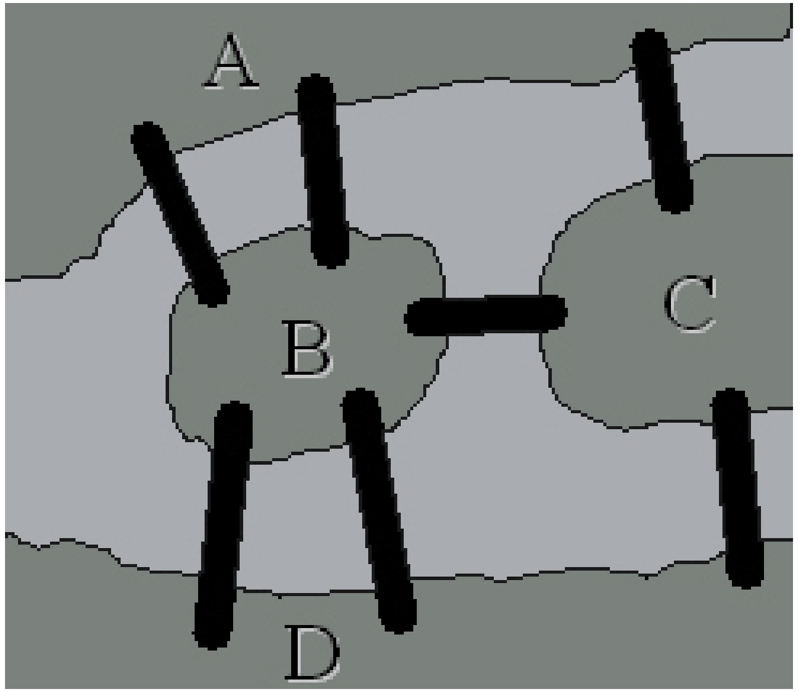
Graph Modeling

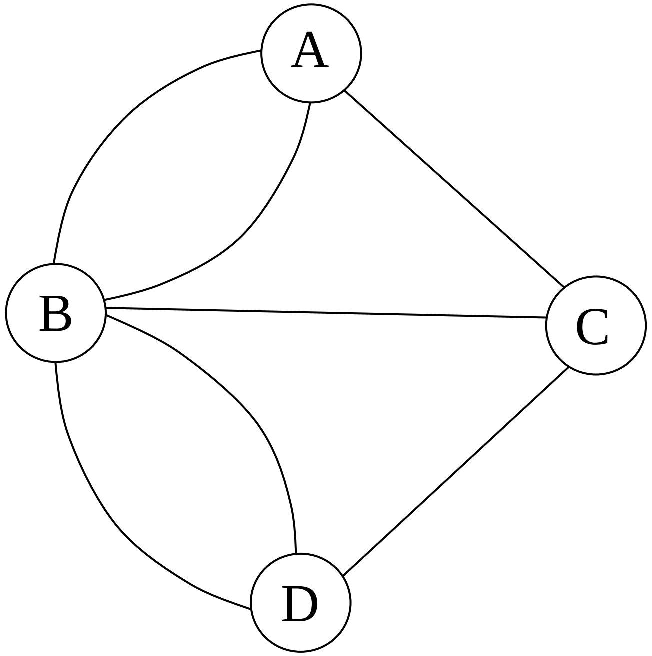


Graph Modeling

Introduction

- **Seven bridges of Königsberg**





Graph Modeling

Škoda Auto University

Basic terminology

- Graph is a set $G = \{V, E\}$, where V is a set of vertices (nodes) and E is a set of edges (arcs).
- Undirected arc is a set of two vertices $\{i, j\}$.
- Directed arc is an ordered pair of two vertices (i, j).
- In undirected graph all arcs are undirected.
- In directed graph (digraph) all arcs are directed.
- Mixed graph contains both undirected and directed arcs.
- Two nodes that are contained in an arc are adjacent.
- Two arcs that share a node are adjacent.
- An arc and a node contained in that arc are incident.
- Degree of a node (in undirected graph) is a number of incident arcs.

Graph Modeling

Škoda Auto University

Basic terminology

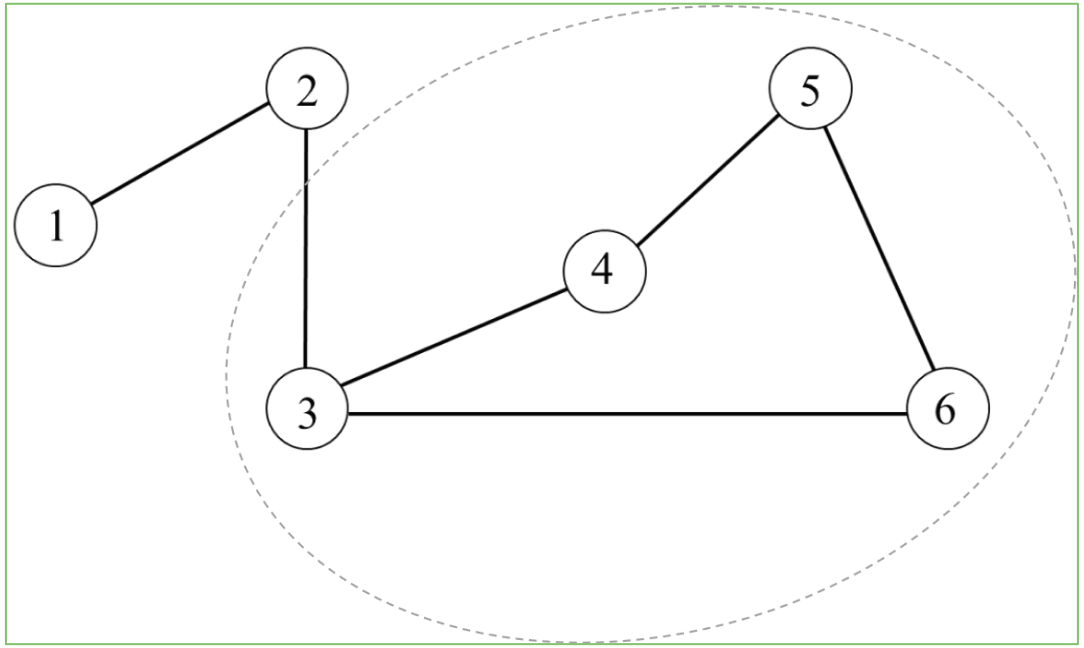
- In-degree of a node (in directed graph) is a number of incident arcs in which the node is the terminal one.
- Out-degree of a node (in directed graph) is a number of incident arcs in which the node is the initial one.
- Walk from node i to node j is a sequence of nodes and arcs, where i is the initial node and j is the terminal node (nodes and arcs may be repeated).
- Trail is a walk with no repeated arc.
- Path is a trail with no repeated node.
- Cycle is closed walk (the initial node is the terminal one).
- In directed path (in directed graph) a direction of all arcs is respected.
- In undirected path (in directed graph) a direction of all arcs may not be respected.

Graph Modeling

Škoda Auto University

Basic terminology

- Cycle (circuit)



Graph Modeling

Škoda Auto University

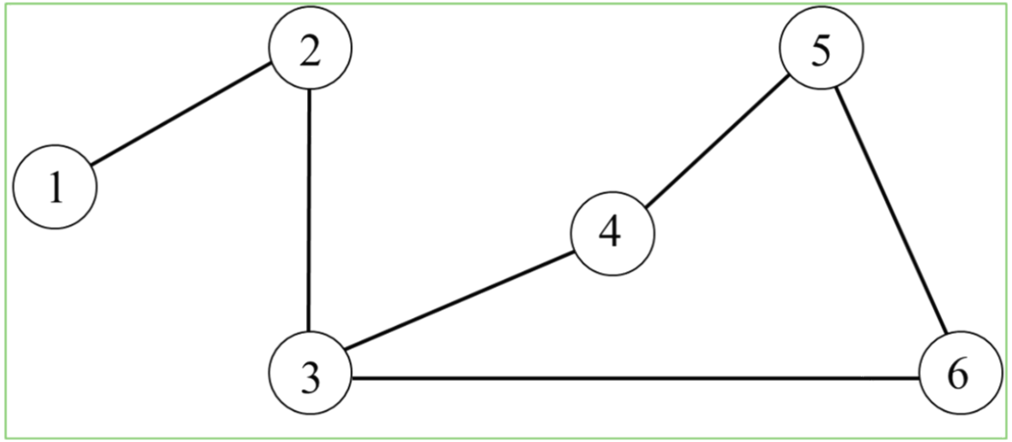
Basic terminology

- Undirected graph is connected if between each pair of nodes there is a path.
- Directed graph is connected if there is a directed or undirected path between each pair of nodes.
- Directed graph is strongly connected if there is a directed path between each pair of nodes.
- Graph is complete if there is an arc between each pair of nodes.
- Tree is a connected undirected graph with no cycles.
- Subgraph of graph $G = \{V, E\}$ is a graph $G' = \{V', E'\}$, where $V' \subseteq V$ and $E' \subseteq E$.
- Spanning tree of the graph G is a subgraph G', where $V' = V$, and which is a tree.
- Valued graph has numbers associated with nodes or/and arcs.
- The minimum spanning tree of a graph is the tree with the minimum sum of edge ratings.

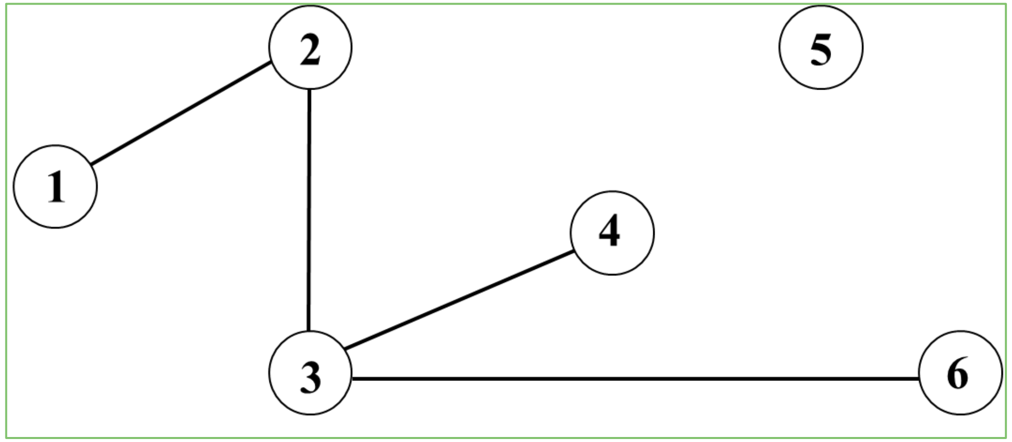
Graph Modeling

Basic terminology

- Connected graph



- Unconnected graph

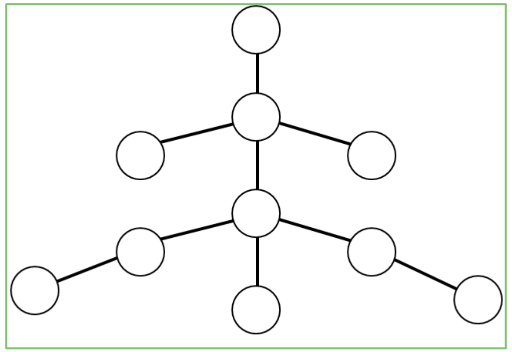
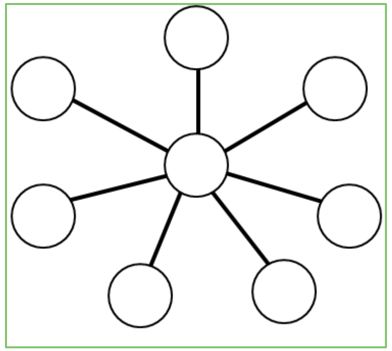
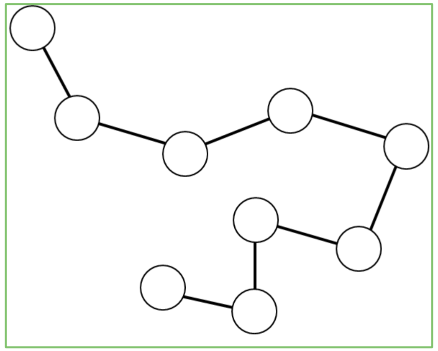


Graph Modeling

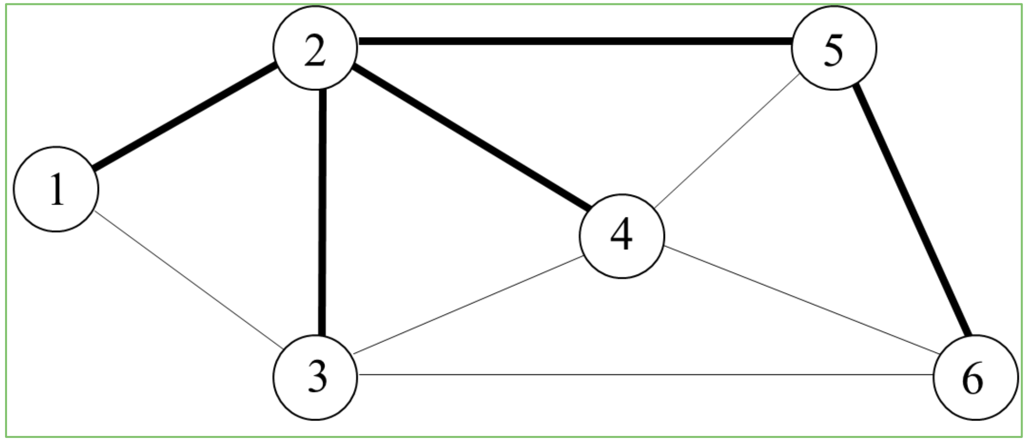
Škoda Auto University

Basic terminology

- Tree

- Spanning tree



Graph Modeling

Škoda Auto University

Basic terminology

- A net graph is a continuous, oriented, valued graph with one input and one output.
- Hamiltonian cycle is a cycle that includes each node of the graph exactly once.
- Eulerian cycle includes each arc of the graph exactly once.
- Eulerian trail is a trail that includes each arc of the graph.
- Eulerian graph is a graph in which the Eulerian cycle can be found.

Graph Modeling

Maximum Flow Problem

- **Definition of the problem**
  - $G = \{V, E\}$ be a digraph with the flow capacity k_{ij} given for each arc (i, j).
  - The objective is to identify the maximum amount of flow that can occur from source node s to sink node d.

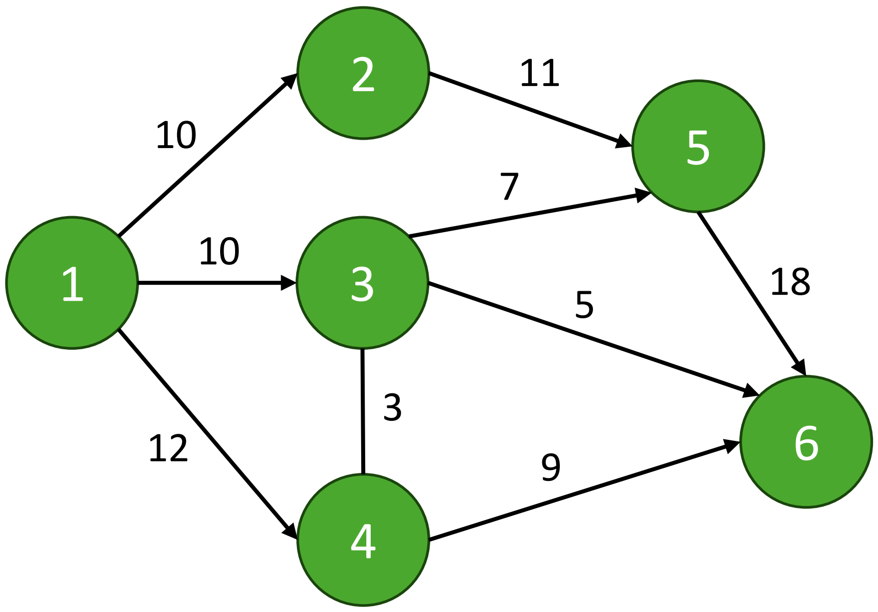
Graph Modeling

Škoda Auto University

Maximum Flow Problem

- **Example**
  - Will you find the maximum flow from node 1 to node 6 for a graph given by the following table.

Arc	Capacity	Arc	Capacity
(1,2)	10	(3,5)	7
(1,3)	10	(3,6)	5
(1,4)	12	(4,3)	3
(2,5)	11	(4,6)	9
(3,4)	3	(5,6)	18



- **Transformation of the grpah**
  - Graph is transformed into complete digraph.
  - Capacities for non-existing arcs are zero.

Graph Modeling

Maximum Flow Problem – mathematical model formulation

- **Decision variables**

$x_{ij} =$ flow from node i to node j $(i, j = 1, 2, \ldots, n)$

$F =$ total flow value

- **Model**

$$F \to \max,$$

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = \begin{cases} F & i = s \\ -F & i = d, \\ 0 & i = 1, 2, \ldots, n, i \notin \{s, d\} \end{cases}$$

$$x_{ij} \le k_{ij} \quad i, j = 1, 2, \ldots, n,$$

$$x_{ij} \in \mathbf{R}_+ \quad i, j = 1, 2, \ldots, n,$$

$$F \in \mathbf{R}_+.$$

```
TITLE MaximalFlow;

OPTIONS
EXCELWORKBOOK="MaximalFlow.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("node");
j:=i;
s[i]:=EXCELRANGE("s");
d[i]:=EXCELRANGE("d");
transitNodes[i]:=i-s-d;

DATA
k[i,j]:=EXCELRANGE("capacity");

VARIABLES
x[i,j] EXPORT TO EXCELRANGE("flow");
F EXPORT TO EXCELRANGE("total");

MODEL
MAX F;

SUBJECT TO
source[i in s]:               sum(j:x[i,j])-sum(j:x[i:=j,j:=i])=F;
destination[i in d]:          sum(j:x[i,j])-sum(j:x[i:=j,j:=i])=-F;
transit[i in transitNodes]:   sum(j:x[i,j])-sum(j:x[i:=j,j:=i])=0;

BOUNDS
x<=k;

END
```

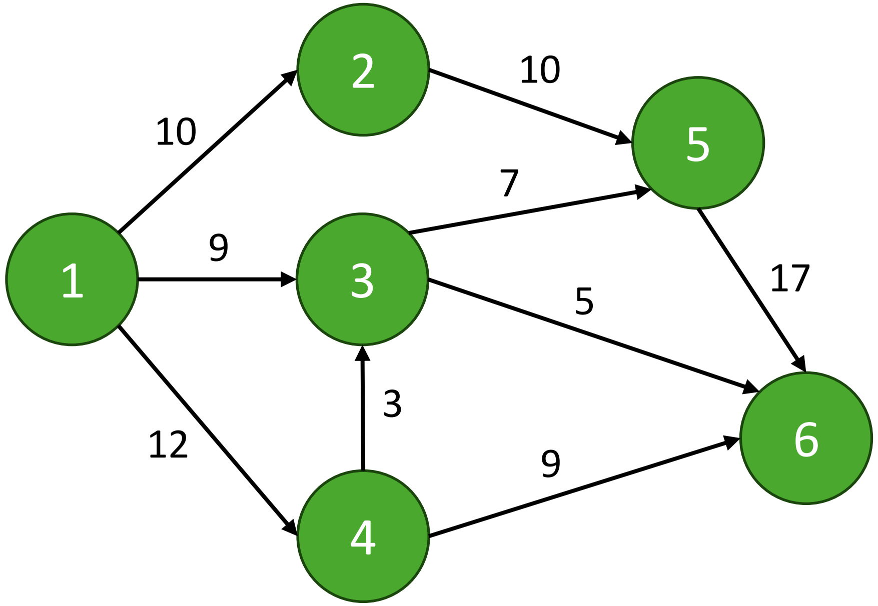
Graph Modeling

Maximum Flow Problem – solution

- **Optimal solution**
  - Decision variables

    see the picture
  - Objective value

    $z_0 = 31$ (maximum possible flow)



SOLUTION RESULT

Optimal solution found

MAX F = 31.0000

DECISION VARIABLES

VARIABLE x[i,j] :

i	j	Activity	Reduced Cost
1	2	10.0000	1.0000
1	3	9.0000	0.0000
1	4	12.0000	0.0000
2	5	10.0000	0.0000
3	5	7.0000	1.0000
3	6	5.0000	1.0000
4	3	3.0000	0.0000
4	6	9.0000	1.0000
5	6	17.0000	0.0000

Graph Modeling

Maximum Flow Problem

- **Alternative approach to modelling**
  - In complete digraph we set capacity $k_{ds} = M$ (big number).
- **Decision variables**

$$x_{ij} = \text{ flow from node } i \text{ to node } j \ (i, j = 1, 2, \ldots, n)$$

- **General model**

$$x_{ds} \to \max,$$

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = 0 \qquad (i = 1, 2, \ldots, n),$$

$$x_{ij} \le k_{ij} \qquad (i, j = 1, 2, \ldots, n),$$

$$x_{ij} \in \mathbf{R}_{+} \qquad (i, j = 1, 2, \ldots, n).$$

9		1	2	3	4	5	6	From node
10	1	0	10	10	11	0	0	31
11	2	0	0	0	0	10	0	10
12	3	0	0	0	0	7	5	12
13	4	0	0	2	0	0	9	11
14	5	0	0	0	0	0	17	17
15	6	31	0	0	0	0	0	31
16	To node	31	10	12	11	17	31	
17								
18	Total	31						

Graph Modeling

Minimum-Cost Flow Problem

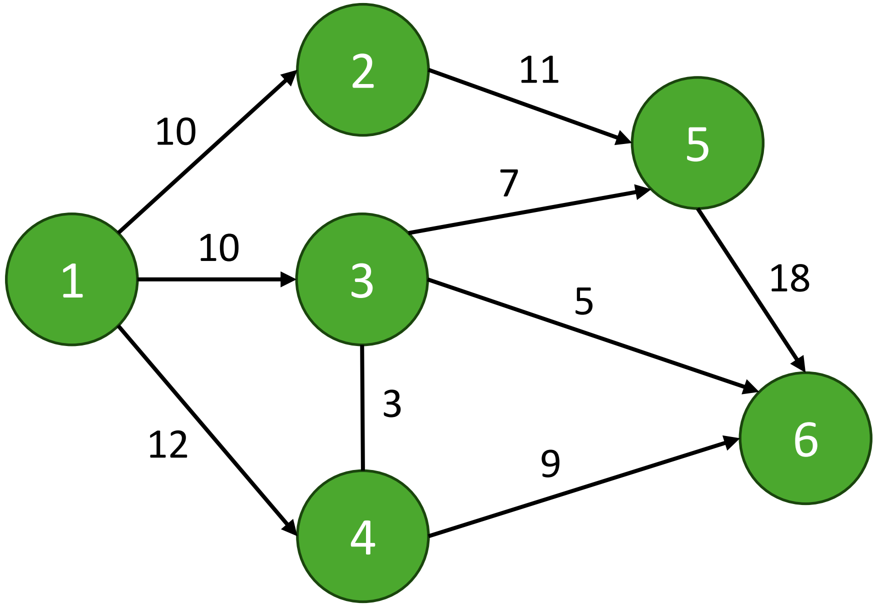
- **Definition of the problem**
  - $G = \{V, E\}$ is a digraph with the flow capacity k_{ij} and unit cost c_{ij} given for each arc (i, j).
  - The objective is to satisfy required total flow F_0 (from source node s to sink node d) with the minimum total cost.
- **Transformation of the grpah**
  - Graph is transformed into complete digraph.
  - Capacities for non-existing arcs are zero.

Graph Modeling

Minimum-Cost Flow Problem

- **Example**
  - Will you find the flow (from 1 to 6) of value 25 with the minimal total cost. In the table, capacity and unit cost for each arc are given.

Arc	Capacity	Cost	Arc	Capacity	Cost
(1,2)	10	5	(3,5)	7	6
(1,3)	10	10	(3,6)	5	9
(1,4)	12	20	(4,3)	3	12
(2,5)	11	11	(4,6)	9	17
(3,4)	3	12	(5,6)	18	8



Graph Modeling

Škoda Auto University

Minimum-Cost Flow Problem – mathematical model formulation

- **Decision variables**

  $x_{ij} = $ flow from node i to node j $(i, j = 1, 2, \ldots, n)$

- **Model**

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \rightarrow \min,$$

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = \begin{cases} F_0 & i = s \\ -F_0 & i = d, \\ 0 & i = 1, 2, \ldots, n, i \notin \{s, d\} \end{cases}$$

$$x_{ij} \leq k_{ij} \quad i, j = 1, 2, \ldots, n,$$

$$x_{ij} \in \mathbf{R}_+ \quad i, j = 1, 2, \ldots, n.$$

```
TITLE FlowMinimalCost;

OPTIONS
EXCELWORKBOOK="FlowMinimalCost.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("node");
j:=i;
s[i]:=EXCELRANGE("s");
d[i]:=EXCELRANGE("d");
transitNodes[i]:=i-s-d;

DATA
k[i,j]:=EXCELRANGE("capacity");
c[i,j]:=EXCELRANGE("cost");
F0:=EXCELRANGE("requirement");

VARIABLES
x[i,j] EXPORT TO EXCELRANGE("flow");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x) ;

SUBJECT TO
source[i in s]:                 sum(j:x[i,j])-sum(j:x[i:=j,j:=i])=F0;
destination[i in d]:            sum(j:x[i,j])-sum(j:x[i:=j,j:=i])=-F0;
transit[i in transitNodes]:     sum(j:x[i,j])-sum(j:x[i:=j,j:=i])=0;

BOUNDS
x<=k;

END
```

Graph Modeling

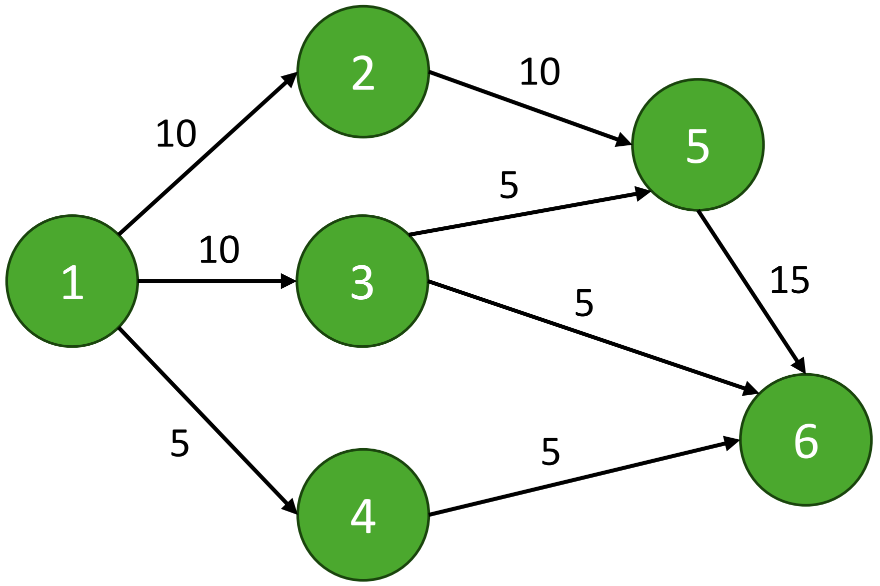
Škoda Auto University

Minimum-Cost Flow Problem – solution

- **Optimal solution**
  - Decision variables

    see the picture
  - Objective value

    $z_0 = 640$ (minimal total cost)




```
SOLUTION RESULT

  Optimal solution found

    MIN z                    =         640.0000

DECISION VARIABLES

VARIABLE x[i,j] :

    i  j             Activity       Reduced Cost
  ---------------------------------------------
   1  2             10.0000           -13.0000
   1  3             10.0000           -13.0000
   1  4              5.0000             0.0000
   2  5             10.0000             0.0000
   3  5              5.0000             0.0000
   3  6              5.0000            -5.0000
   4  6              5.0000             0.0000
   5  6             15.0000             0.0000
  ---------------------------------------------
```

Graph Modeling

Škoda Auto University

Transshipment Problem

- **Definition of the problem**
  - $G = \{V, E\}$ is a digraph with three sets of nodes: set of sources V_s, set of destinations V_d and set of transshipment nodes V_t.
  - Flow capacity k_{ij} and unit cost c_{ij} are given for each arc (i, j).
  - Demand in all destinations has to be satisfied without exceeding any supply.
  - The objective is to minimize total flow cost. Suppose total demand is equal to total supply.

$a_i > 0$ a supply of the product in source node $i \in V_s$

$a_i < 0$ a demand for the product in destination $i \in V_d$

$a_i = 0$ for each transshipment node $i \in V_t$

- **Assumptions**

$$V = V_s \cup V_d \cup V_t \quad \text{a} \quad V_s \cap V_d \cap V_t = \emptyset, \qquad \sum_{i \in V_s} a_i + \sum_{i \in V_d} a_i = 0.$$

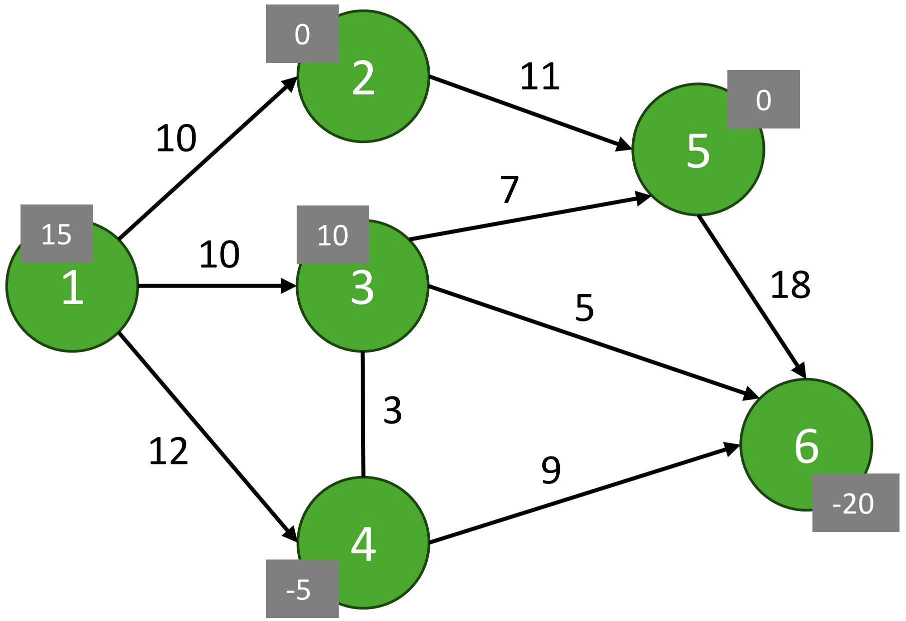
Graph Modeling

Škoda Auto University

Transshipment Problem

- **Example**
  - It is necessary to transport empty containers from sources to destinations.
  - In the graph, nodes 1 and 3 are sources with supply 15 and 10 containers, nodes 4 and 6 are destinations with demand 5 and 20 containers.
  - The table gives the capacity for each arc and the cost of transporting one container.
  - The objective is to minimize total cost.

Arc	Capacity	Cost	Arc	Capacity	Cost
(1,2)	10	5	(3,5)	7	6
(1,3)	10	10	(3,6)	5	9
(1,4)	12	20	(4,3)	3	12
(2,5)	11	11	(4,6)	9	17
(3,4)	3	12	(5,6)	18	8



Graph Modeling

Škoda Auto University

Transshipment Problem – mathematical model formulation

- **Transformation of the grpah**
  - Graph is transformed into complete digraph.
  - Capacities for non-existing arcs are zero.
- **Decision variables**

  x_{ij} = value of the transport from node i to node j $(i, j = 1, 2, \ldots, n)$
- **Model**

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \to \min,$$

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = a_i \qquad i = 1, 2, \ldots, n,$$

$$x_{ij} \le k_{ij} \quad i, j = 1, 2, \ldots, n,$$

$$x_{ij} \in \mathbf{R}_+ \quad i, j = 1, 2, \ldots, n.$$

```
TITLE Transshipment;

OPTIONS
EXCELWORKBOOK="Transshipment.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("node");
j:=i;

DATA
k[i,j]:=EXCELRANGE("capacity");
c[i,j]:=EXCELRANGE("cost");
a[i]:=EXCELRANGE("containers");

INTEGER VARIABLES
x[i,j] EXPORT TO EXCELRANGE("flow");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x) ;

SUBJECT TO
node[i]: sum(j:x[i,j])-sum(j:x[i:=j,j:=i])=a[i];

BOUNDS
x<=k;

END
```

Graph Modeling

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Transshipment Problem – solution

- **Optimal solution**
  - Decision variables

    see the picture
  - Objective value

    $z_0 = 455$ (minimal total cost)

```
SOLUTION RESULT

  Optimal integer solution found

    MIN z                  =        455.0000

DECISION VARIABLES

VARIABLE x[i,j] :

    i  j          Activity     Reduced Cost
  -----------------------------------------
   1  2            8.0000           5.0000
   1  3            2.0000          10.0000
   1  4            5.0000          20.0000
   2  5            8.0000          11.0000
   3  5            7.0000           6.0000
   3  6            5.0000           9.0000
   5  6           15.0000           8.0000
  -----------------------------------------
```

Graph Modeling

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Minimal Spanning Tree

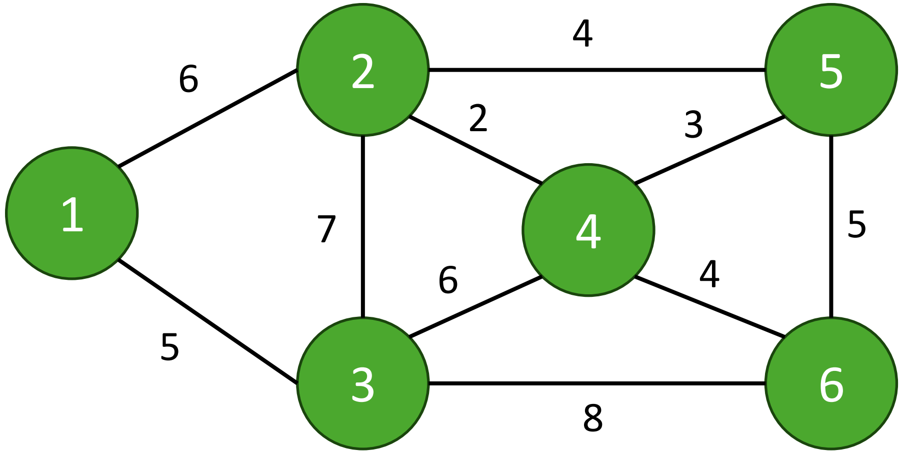
- **Definition of the problem**
  - $G = \{V, E\}$ is an undirected graph with cost c_{ij} given for each arc $\{i, j\}$.
  - The objective is to search a spanning tree of G minimizing total cost.
- **Graph transformation**
  - Set of undirected arcs E is transformed to set of directed arcs A in the following way:
    - Each arc $\{i, j\} \in E$ is replaced with directed arcs $(i, j) \in A$ a $(j, i) \in A$, $c_{ij} = c_{ji}$.

Graph Modeling

Minimal Spanning Tree

- **Example**
  - The company has to install 6 information boards in the city park. They must be connected by a cable placed under pavements.
  - Distances (in ten meters) between boards can be found in the table. If there is no pavement between a pair of boards, prohibitive value 100 is set.
  - The objective is to minimize total cost both on excavation job and on cable itself.

Boards	1	2	3	4	5	6
1	0	6	5	100	100	100
2	6	0	7	2	4	100
3	5	7	0	6	100	8
4	100	2	6	0	3	4
5	100	4	100	3	0	5
6	100	100	8	4	5	0



Graph Modeling

Škoda Auto University

Minimal Spanning Tree – mathematical model formulation

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i, j) \text{ is selected} \\ 0 & \text{otherwise} \end{cases} \qquad (i, j = 1, 2, ..., n)$$

$$y_{ij} = \text{ flow from node } i \text{ to node } j \qquad (i, j = 1, 2, ..., n)$$

- **Model**

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \to \min,$$

$$\sum_{j=1}^{n} x_{1j} = 0, \qquad \sum_{j=1}^{n} x_{ij} = 1 \qquad i = 2, 3, \ldots, n,$$

$$\sum_{j=1}^{n} y_{ij} - \sum_{j=1}^{n} y_{ji} = 1 \qquad i = 2, 3, \ldots, n,$$

$$y_{ij} \le (n-1)\, x_{ij} \qquad i, j = 1, 2, \ldots, n,$$

$$x_{ij} \in \mathbf{B} \qquad i, j = 1, 2, \ldots, n,$$

$$y_{ij} \in \mathbf{R}_{+} \qquad i, j = 1, 2, \ldots n.$$

```
TITLE MinimalSpanningTree;

OPTIONS
EXCELWORKBOOK="MinimalSpanningTree.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("node");
j:=i;

DATA
c[i,j]:=EXCELRANGE("distance");
n:=count(i);

BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("cable");

VARIABLES
y[i,j] EXPORT TO EXCELRANGE("flow");

MODEL

MIN z EXPORT TO EXCELRANGE("total") =sum(c*x) ;

SUBJECT TO
formNode1:            sum(j:x[i:=1,j])=0;
Cycles[i>1]:          sum(j:x[i,j])=1;
Connected[i>1]:       sum(j:y[i,j])-sum(j:y[i:=j,j:=i])=1;
FlowBalance[i,j]:     y<=x*(n-1);

END
```

Graph Modeling

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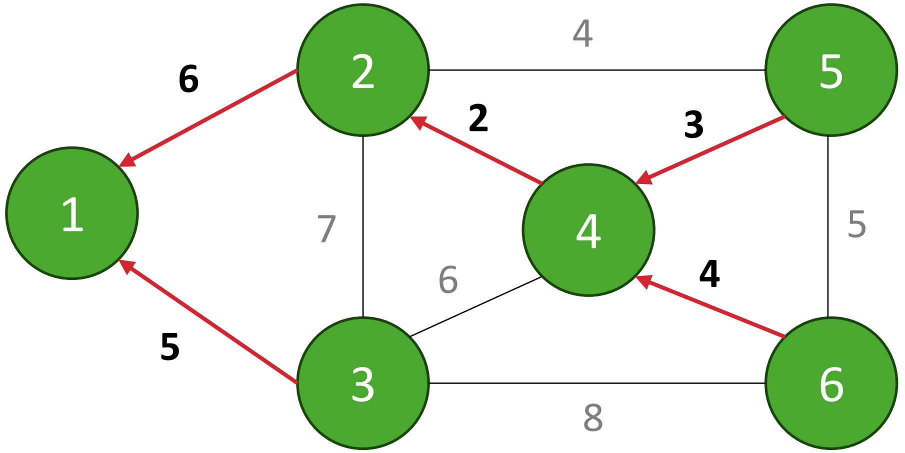
Minimal Spanning Tree – solution

- **Optimal solution**
  - Objective value

$z_0 = 20$ (minimum cable length)



Boards	1	2	3	4	5	6
1	0	6	5	100	100	100
2	6	0	7	2	4	100
3	5	7	0	6	100	8
4	100	2	6	0	3	4
5	100	4	100	3	0	5
6	100	100	8	4	5	0

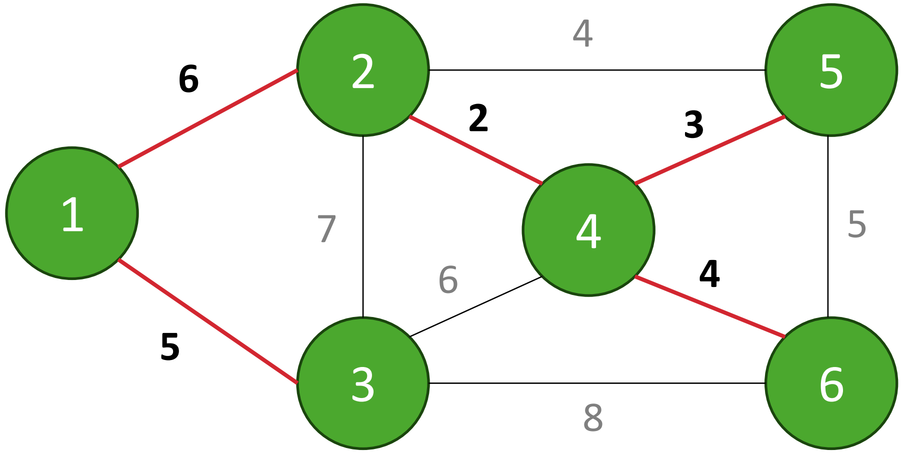
```
SOLUTION RESULT

  Optimal integer solution found

    MIN z                =          20.0000

DECISION VARIABLES

VARIABLE x[i,j] :

     i  j             Activity        Reduced Cost
  -------------------------------------------------
   2  1              1.0000              6.0000
   3  1              1.0000              5.0000
   4  2              1.0000              2.0000
   5  4              1.0000              3.0000
   6  4              1.0000              4.0000
  -------------------------------------------------

VARIABLE y[i,j] :

     i  j             Activity        Reduced Cost
  -------------------------------------------------
   2  1              4.0000              0.0000
   3  1              1.0000              0.0000
   4  2              3.0000              0.0000
   5  4              1.0000              0.0000
   6  4              1.0000              0.0000
  -------------------------------------------------
```

Graph Modeling

Minimal Spanning Tree – solution

- **Optimal solution**
  - Objective value

$z_0 = 20$ (minimum cable length)

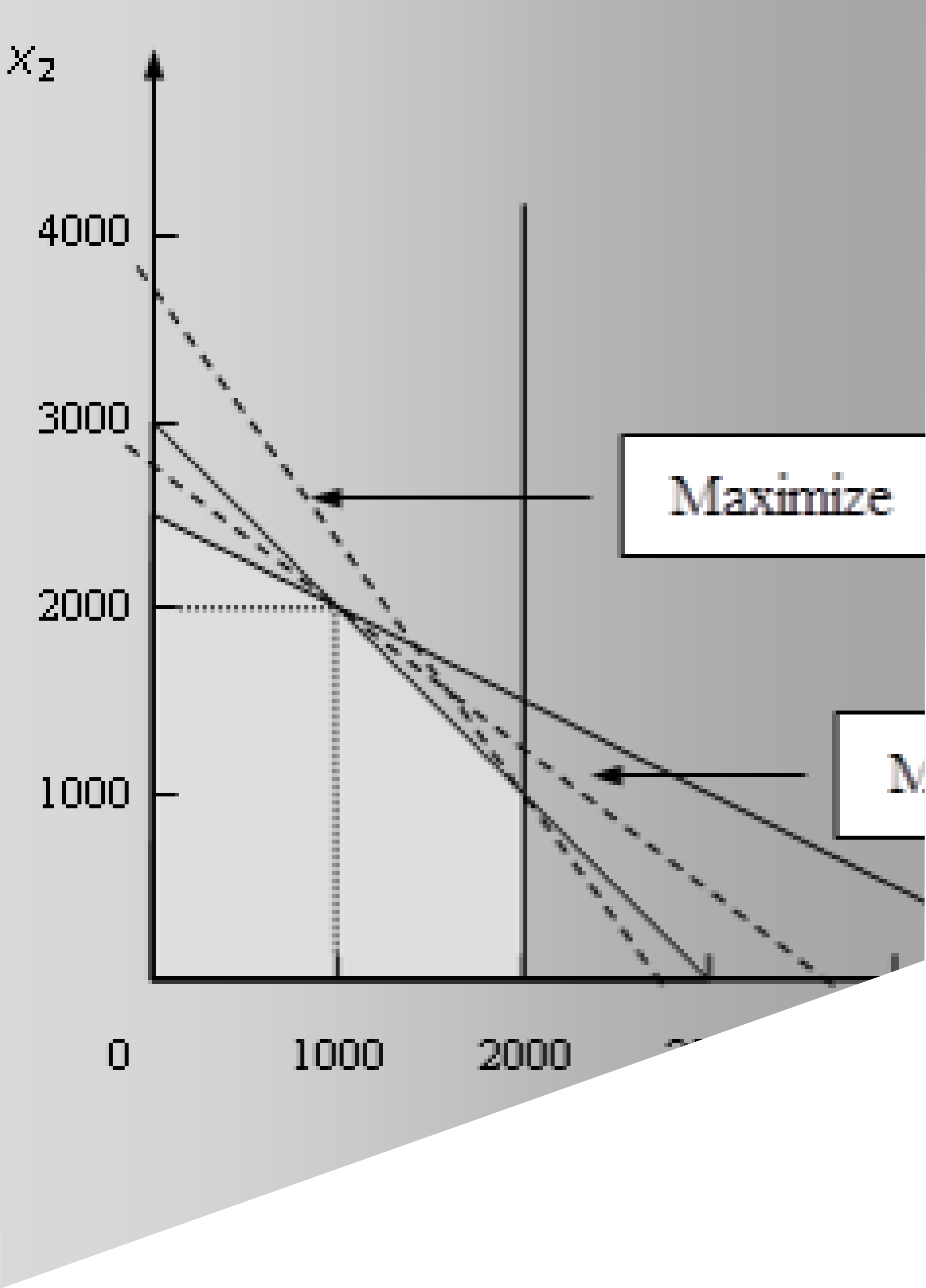


Boards	1	2	3	4	5	6
1	0	6	5	100	100	100
2	6	0	7	2	4	100
3	5	7	0	6	100	8
4	100	2	6	0	3	4
5	100	4	100	3	0	5
6	100	100	8	4	5	0

```
SOLUTION RESULT

  Optimal integer solution found

    MIN z                    =          20.0000

DECISION VARIABLES

VARIABLE x[i,j] :

    i  j              Activity         Reduced Cost
  ------------------------------------------------
   2  1               1.0000              6.0000
   3  1               1.0000              5.0000
   4  2               1.0000              2.0000
   5  4               1.0000              3.0000
   6  4               1.0000              4.0000
  ------------------------------------------------

VARIABLE y[i,j] :

    i  j              Activity         Reduced Cost
  ------------------------------------------------
   2  1               4.0000              0.0000
   3  1               1.0000              0.0000
   4  2               3.0000              0.0000
   5  4               1.0000              0.0000
   6  4               1.0000              0.0000
  ------------------------------------------------
```

5

Transportation and Routing Problems



Transportation and Routing Problems

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Transportation Problem

- **Definition of the problem**
  - Transport of homogeneous product.
  - Set of sources with limited supply.
  - Set of destinations with demand (requirement).
  - Unit shipping cost for all pairs of sources and destinations.
  - The goal is to satisfy all requirements without exceeding any supply.
  - The objective is to find shipments to minimize total shipping cost.
- **Type of the problem**
  - Balanced – total supply is equal to total demand.
  - Unbalanced – total supply is different from total demand, it is possible to make the problem balanced:
    - adding dummy destination,
    - finding additional source or adding dummy source (with the possibility of unsatisfied requirement).

Transportation and Routing Problems

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Transportation Problem

- **Example**
  - A company producing petroleum products is establishing four new gas stations in Tábor, Příbram, Jindřichův Hradec and Písek
  - The product is gasoline, which will be imported from warehouses in Pilsen, České Budějovice and Jihlava.

Transportation and Routing Problems

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Transportation Problem

- **Example**
  - The table shows the weekly supply of warehouses and the planned weekly requirements of filling stations (**in hectoliters**). Gasoline will be transported by road (once a week). The table contains unit sipping costs for transporting **one hectoliter** of gasoline from sources to destinations (in CZK).
  - The objective is to plan the transportation of gasoline so that the total shipping costs are minimal. This shipping schedule, of course, must satisfy requirement of each destination, and must not exceed supply of any warehouse.

Source \ Destination	Tábor	Příbram	Jindřichův Hradec	Písek	Supply (hectoliter)
Plzeň	10	8	20	9	110
České Budějovice	9	13	6	13	160
Jihlava	7	11	10	18	180
Demand (hectoliters)	90	130	80	120	

Transportation and Routing Problems

Transportation Problem

- **Example**
  - Balancing Transportation Problem by adding dummy customer.

Source \ Destination	Tábor	Příbram	Jindřichův Hradec	Písek	Dummy destination	Supply (hectoliter)
Plzeň	10	8	20	9	0	110
České Budějovice	9	13	6	13	0	160
Jihlava	7	11	10	18	0	180
Demand (hectoliters)	90	130	80	120	30	

Transportation and Routing Problems

Transportation Problem – feasible solution

- **North-West Corner Method**
  1. Select the upper left-hand corner cell of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand, i.e., min(s1, d1).
  2. Adjust the supply and demand numbers in the respective rows and columns.
  3. If the demand for the first cell is satisfied, then move horizontally to the next cell in the second column.
  4. If the supply for the first row is exhausted, then move down to the first cell in the second row.
  5. If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.
  6. Continue the process until all supply and demand values are exhausted.

Source \ Destination	Tábor	Příbram	Jindřichův Hradec	Písek	Dummy destination	Supply (hectoliter)
Plzeň	90	20	0	0	0	110
České Budějovice	0	110	50	0	0	160
Jihlava	0	0	30	120	30	180
Demand (hectoliters)	90	130	80	120	30	

- Objective value

$$z = 5250$$

Transportation and Routing Problems

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Transportation Problem – feasible solution

- **Matrix Minimum Method**
  1. Find a minimal cost for all possible shipments.
  2. Assign the shipment to the pair od source and destination with the minimal cost found in step 1. The value of the shipment is equal to the minimum of remaining supply and remaining demand for this pair.
  3. Decrease remaining supply and remaining demand, for the pair of the source and destination, by the shipment calculated in step 2.
  4. If there is some remaining demand go to step 1, otherwise, the feasible solution is found.

Source \ Destination	Tábor	Příbram	Jindřichův Hradec	Písek	Dummy destination	Supply (hectoliter)
Plzeň	0	110	0	0	0	110
České Budějovice	0	0	80	80	0	160
Jihlava	90	20	0	40	30	180
Demand (hectoliters)	90	130	80	120	30	

- Objective value

$$z = 3970$$

Transportation and Routing Problems

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Transportation Problem

- **General mathematical model**

$$z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \rightarrow \min,$$

$$\sum_{j=1}^{n} x_{ij} \le a_i \qquad i = 1, 2, \dots, m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j = 1, 2, \dots, n,$$

$$x_{ij} \in \mathbf{R}_+ \qquad i = 1, 2, \dots, m;\ j = 1, 2, \dots, n.$$

```
TITLE TransportationProblem;

OPTIONS
EXCELWORKBOOK="TP.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("source");
j:=EXCELRANGE("destination");

DATA
a[i]:=EXCELRANGE("supply");
b[j]:=EXCELRANGE("demand");
c[i,j]:=EXCELRANGE("cost");

VARIABLES
x[i,j] EXPORT TO EXCELRANGE("shipment");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x)

SUBJECT TO

source[i]:       sum(j:x[i,j])<=a[i];
destination[j]:  sum(i:x[i,j])=b[j];

END
```

Transportation and Routing Problems

Škoda Auto University

Transportation Problem – optimal solution

- **Optimal solution**
  - Decision variables

Source \ Destination	Tábor	Příbram	Jindřichův Hradec	Písek	Supply (hectoliter)
Plzeň	0	40	0	70	110
České Budějovice	0	0	80	50	160
Jihlava	90	90	0	0	180
Demand (hectoliters)	90	130	80	120	

  - Objective value

    $z_0 = 3700$ (minimum total transport costs)

  - Slack/surplus variables

    $v_2 = 30$ (will remain in the warehouse in České Budějovice)

Transportation and Routing Problems

Škoda Auto University

Container Transportation Problem

- Based on the transportation problem.
- Goods is transported in containers of the same capacity.
- The shipping costs are not associated with the transported unit, but with the use of one container between the source and the destination.
- The objective is to determine the shipments between sources and destinations and the number of containers used for the transport at the minimal total transportation costs.

Transportation and Routing Problems

Container Transportation Problem

- **Example**
  - Let's assume that in the previous example, the transport company will charge prices for the transport (rental) of one tank between individual sources and destinations.
  - Tanks with a capacity of 20 hl can be used for transport.
  - The objective is to determine how many hectoliters of gasoline will be shipped between the individual locations, but in addition, to determine how many tanks will be used for this transport so that the total shipping costs are minimal.

Source \ Destination	Tábor	Příbram	Jindřichův Hradec	Písek	Supply (hectoliter)
Plzeň	210	130	420	190	110
České Budějovice	170	260	130	260	160
Jihlava	150	210	180	340	180
Demand (hectoliters)	90	130	80	120	

Transportation and Routing Problems

Container Transportation Problem – mathematical model formulation

- Mathematical model is based on the mathematical model of transportation problem.
- **Decision variables**

$$x_{ij} = \text{ the shipment } \left(\text{in tuns or hl}\right) \text{ from } i\text{-th supplier to } j\text{-th customer}$$

$$y_{ij} = \text{ number of containers used for transport from } i\text{-th supplier to } j\text{-th customer}$$

- **Objective**

$$z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} y_{ij} \rightarrow \min$$

- **Constraints**
  - It is necessary to add following constraints to the constraints of transportation problem:

$$x_{ij} \le K y_{ij} \qquad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n,$$

$$y_{ij} \in \mathbf{Z}_{+} \qquad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n.$$

```
TITLE ContainerTransportationProblem;

OPTIONS
EXCELWORKBOOK="CTP.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("source");
j:=EXCELRANGE("destination");

DATA
a[i]:=EXCELRANGE("supply");
b[j]:=EXCELRANGE("demand");
c[i,j]:=EXCELRANGE("cost");
K:=EXCELRANGE("capacity");

VARIABLES
x[i,j] EXPORT TO EXCELRANGE("shipment");

INTEGER VARIABLES
y[i,j] EXPORT TO EXCELRANGE("tank");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*y)

SUBJECT TO

source[i]:       sum(j:x[i,j])<=a[i];
destination[j]:  sum(i:x[i,j])=b[j];
tanks[i,j]:      x<=K*y;

END
```

Transportation and Routing Problems

Škoda Auto University

Container Transportation Problem – optimal solution

- **Optimal solution**
  - Decision variables (transferred volume)

Source \ Destination	Tábor	Příbram	Jindřichův Hradec	Písek	Supply (hectoliter)
Plzeň	0	70	0	40	110
České Budějovice	0	0	80	80	160
Jihlava	90	60	0	0	180
Demand (hectoliters)	90	130	80	120	

  - Objective value

    $z_0 = 3840$ (minimum total transport costs)

  - Slack/surplus variables

    $v_3 = 30$ (will remain in the warehouse in Jihlava)

Transportation and Routing Problems

Container Transportation Problem – optimal solution

- **Optimal solution**
  - Decision variables (number of tankers)

Source \ Destination	Tábor	Příbram	Jindřichův Hradec	Písek	Supply (hectoliter)
Plzeň	0	4	0	2	110
České Budějovice	0	0	4	4	160
Jihlava	5	3	0	0	180
Demand (hectoliters)	90	130	80	120	

Transportation and Routing Problems

Škoda Auto University

Bin Packing Problem

- **Definition of the BPP I**
  - A set of n items that can be packed into m containers.
  - The weight w_j and value c_j of item j are given.
  - Let K_i be a weight capacity of container i.
  - The objective is to maximize the total value of all assigned items.

Transportation and Routing Problems

Škoda Auto University

Bin Packing Problem – BPP I

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if item } j \text{ is assigned to container } i \\ 0 & \text{otherwise} \end{cases}$$

- **Model**

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_j x_{ij} \to \max,$$

$$\sum_{i=1}^{m} x_{ij} \le 1 \qquad j = 1, 2, \ldots, n,$$

$$\sum_{j=1}^{n} w_j x_{ij} \le K_i \qquad i = 1, 2, \ldots, m,$$

$$x_{ij} \in \mathbf{B} \qquad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n.$$

```
TITLE BinPackingProblemI;

OPTIONS
EXCELWORKBOOK="BPPI.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("container");
j:=EXCELRANGE("product");

DATA
w[j]:=EXCELRANGE("weight");
c[j]:=EXCELRANGE("value");
K[i]:=EXCELRANGE("capacity");


BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("placement");

MODEL
MAX z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
product[j]:    sum(i:x[i,j])<=1;
container[i]:  sum(j:w[j]*x[i,j])<K[i];

END
```

Transportation and Routing Problems

Škoda Auto University

Bin Packing Problem (BPP I) – optimal solution

- **Optimal solution**
  - Placement of products in containers

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
1			Product	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12	P13	P14	P15	P16	P17	P18	P19	P20
2			Weight	10	7	6	12	5	9	11	3	12	7	9	13	12	7	5	4	9	8	4	15
3			Value	80	130	80	100	30	70	150	30	140	80	110	130	110	80	70	30	120	100	60	170
4	Container	Capacity																					
5	K1	25		0	0	1	0	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0
6	K2	20		0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
7	K3	28		0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0	0
8	K4	30		0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	0
9	K5	27		0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1
10																							
11	Total	1560																					

  - Weight of container load

4	Container	Capacity	Load
5	K1	25	25
6	K2	20	20
7	K3	28	28
8	K4	30	29
9	K5	27	27

Transportation and Routing Problems

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Bin Packing Problem

- **Definition of the problem 2.**
  - A set of n types of items that have to be transported using m containers of identical weight capacity K.
  - Let w_j be a weight of item type j and r_j be a number of them to be transported.
  - The objective is to minimize a number of containers used to transport all items.

Transportation and Routing Problems

Škoda Auto University

Bin Packing Problem – BPP II

- **Example**
  - Products must be transported to the client using identical containers. In the table, a unit weight of each product type (in kg) and a number of them to transport are given. The weight capacity of the container is 500 kg.
  - The objective is to minimize a number of used containers.

BinPacking	Weight	Number
Product 1	20	13
Product 2	22	15
Product 3	18	25
Product 4	15	30
Product 5	21	18
Product 6	16	35

Transportation and Routing Problems

Škoda Auto University

Bin Packing Problem – BPP II

- **Decision variables**

$$x_i = \begin{cases} 1 & \text{if container } i \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ij} = \text{ a number of items of type } j \text{ being transported in container } i$$

- **Model**

$$z = \sum_{i=1}^{m} x_i \rightarrow \min,$$

$$\sum_{i=1}^{m} y_{ij} = r_j \qquad j = 1, 2, \ldots, n,$$

$$\sum_{j=1}^{n} w_j y_{ij} \le K x_i \qquad i = 1, 2, \ldots, m,$$

$$x_i \in \mathbf{B} \qquad i = 1, 2, \ldots, m,$$

$$y_{ij} \in \mathbf{Z}_+ \qquad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n.$$

```
TITLE BinPackingProblemII;

OPTIONS
EXCELWORKBOOK="BPPII.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("container");
j:=EXCELRANGE("product");

DATA
w[j]:=EXCELRANGE("weight");
r[j]:=EXCELRANGE("number");
K:=EXCELRANGE("capacity");


BINARY VARIABLES
x[i] EXPORT TO EXCELRANGE("used");

INTEGER VARIABLES
y[i,j] EXPORT TO EXCELRANGE("placement");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(x);

SUBJECT TO
product[j]:    sum(i:y[i,j])=r[j];
container[i]:  sum(j:w[j]*y[i,j])<=K*x[i];

END
```

Transportation and Routing Problems

Bin Packing Problem (BPP II) – optimal solution

- **Optimal solution**
  - Objective value

$z_0 = 5$ (minimum number of containers)

	A	B	C	D	E	F	G	H	I
1	Product	P1	P2	P3	P4	P5	P6		Capacity
2	Weight	20	22	18	15	21	16		500
3	Number	13	15	25	30	18	35		
4									
5	Container	Number of Products						Used	Load
6	K1	0	0	0	30	0	0	1	450
7	K2	13	0	0	0	11	0	1	491
8	K3	0	0	25	0	2	0	1	492
9	K4	0	15	0	0	5	4	1	499
10	K5	0	0	0	0	0	31	1	496
11	K6	0	0	0	0	0	0	0	0
12									
13	Total	5							

Transportation and Routing Problems

Shortest Path Problem

- The objective is to find the shortest path between a pair of nodes.
- A task solved on a daily basis when using navigation in a car or searching for a connection on a map portal.
- There are many algorithms that can be used to easily find all distances between all pairs of nodes in a given graph. Dijkstra's algorithm is designed to find the shortest paths from one particular node to all other nodes in the graph. It can be used repeatedly for different default nodes.
- A simple mathematical model can also be used for the solution.

Transportation and Routing Problems

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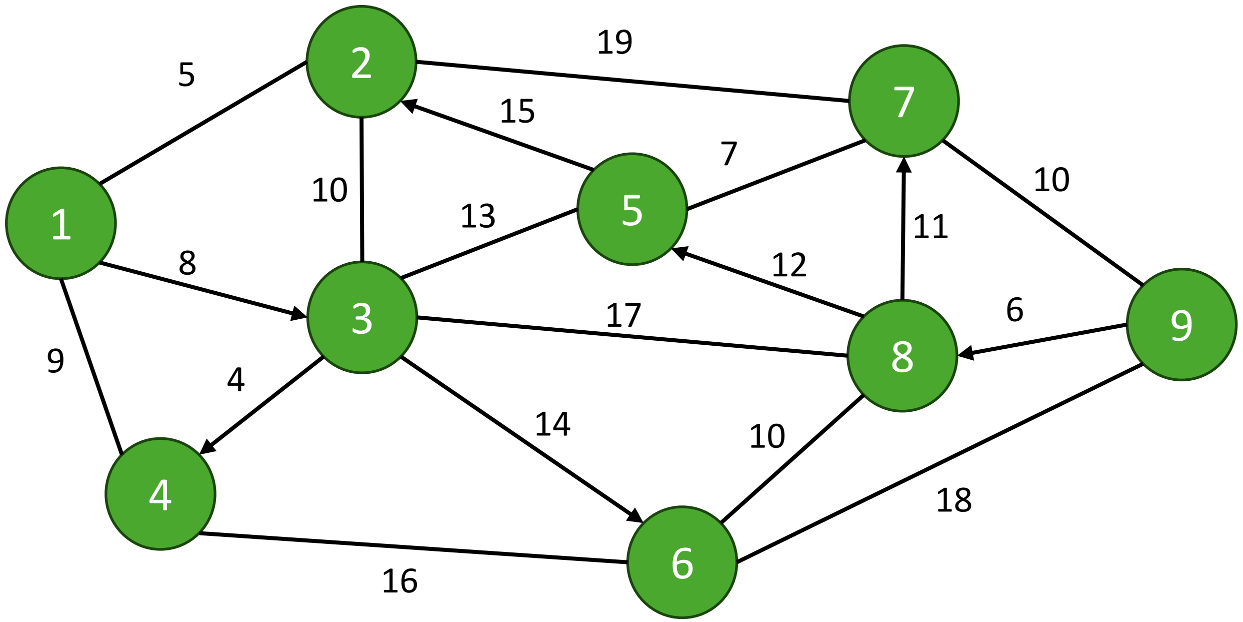
Shortest Path Problem

- **Example**
  - A distribution firm has been awarded a contract to transport an overweight cargo from location 1 to location 9.
  - Distances in kilometers are given for all arcs in the graph. Due to the fact that some roads have sections with dangerous drops, they can only be transported in one direction, which is indicated by oriented arcs.
  - The objective is to travel the minimum distance when transporting cargo.



Transportation and Routing Problems

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Shortest Path Problem

- **Table with lengths of arcs**

Arcs	1	2	3	4	5	6	7	8	9
1	1000	5	8	9	1000	1000	1000	1000	1000
2	5	1000	10	1000	1000	1000	19	1000	1000
3	1000	10	1000	4	13	14	1000	17	1000
4	9	1000	1000	1000	1000	16	1000	1000	1000
5	1000	15	13	1000	1000	1000	7	1000	1000
6	1000	1000	1000	16	1000	1000	1000	10	18
7	1000	19	1000	1000	7	1000	1000	1000	10
8	1000	1000	17	1000	12	10	11	1000	1000
9	1000	1000	1000	1000	1000	18	10	6	1000

Transportation and Routing Problems

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Minimal Path Problem – mathematical model

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if the vehicle goes from place } i \text{ to place } j \\ 0 & \text{otherwise} \end{cases}$$

- **Model**

$$z = \sum_{i=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} \to \min,$$

$$\sum_{i=1}^{n} x_{i1} = 0, \qquad \sum_{j=1}^{n} x_{nj} = 0,$$

$$\sum_{j=1}^{n} x_{1j} = 1, \qquad \sum_{i=1}^{n} x_{in} = 1,$$

$$x_i \in \mathbf{B} \qquad i = 1, 2, \ldots, n; \;\; j = 1, 2, \ldots, n.$$

```
TITLE NejkratsiCesta;

OPTIONS
ExcelWorkBook="NejkratsiCesta.xlsx";
ExcelSheetName="MPL";

INDEX
i:=EXCELRANGE("uzel");
j:=i;
s[i]:=EXCELRANGE("s");
d[i]:=EXCELRANGE("d");
prubezne[i]:= i-s-d;

DATA
c[i,j]:=EXCELRANGE("vzdalenost");

BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("cesta");


MODEL
MIN z EXPORT TO EXCELRANGE("celkem") =sum(c*x);

SUBJECT TO

doUzlus[j in s]:             sum(i:x[i,j])=0;
zUzlud[i in d]:              sum(j:x[i,j])=0;
zUzlus[i in s]:              sum(j:x[i,j])=1;
doUzlud[j in d]:             sum(i:x[i,j])=1;
prubezneUzly[i in prubezne]: sum(j:x[i,j])=sum(j:x[i:=j,j:=i]);

END
```

Transportation and Routing Problems

Škoda Auto University

Shortest Path Problem – optimal solution

- **Optimal solution**
  - Objective value

    $$z_0 = 34$$

  - Arcs in the path

Arcs	1	2	3	4	5	6	7	8	9
1	0	1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0

Transportation and Routing Problems

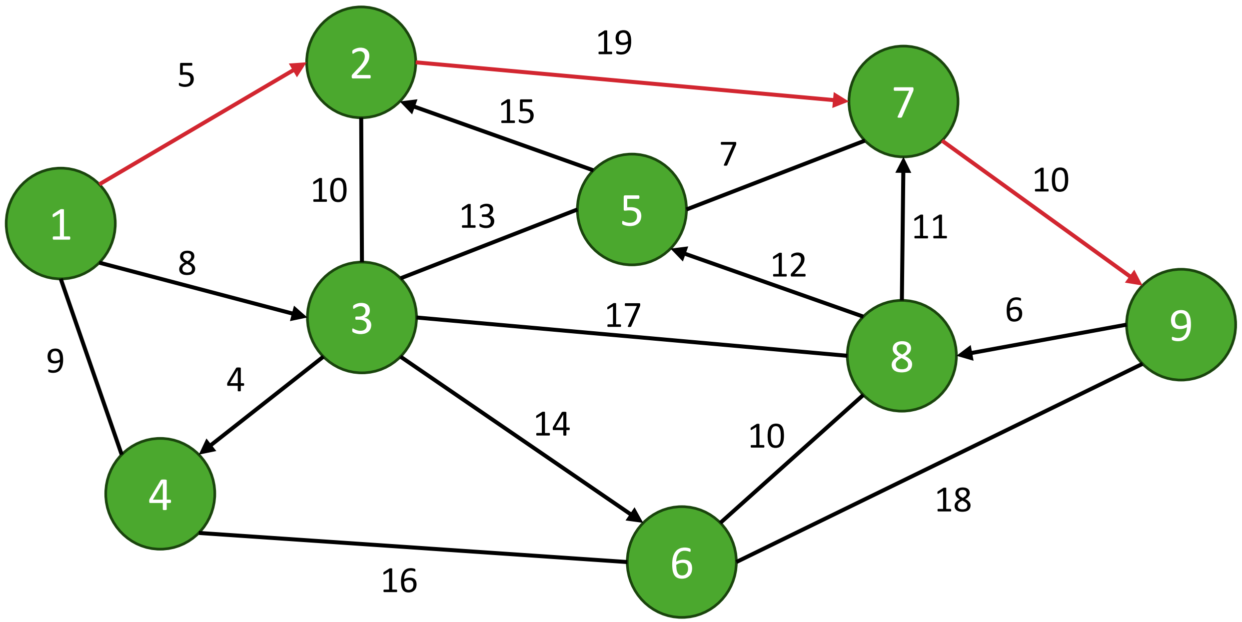
Škoda Auto University

Shortest Path Problem – optimal solution

- **Optimal solution**
  - Objective value

    $$z_0 = 34$$

  - Arcs in the path



Transportation and Routing Problems

Škoda Auto University

Treveling Salesman Problem

- **Definition of the problem**
  - Set of customers.
  - Each customer must be visited exactly once.
  - Cyclical route starts and ends in the home city (index 1).
  - Evaluation of direct travel form location i to location j is denoted by c_{ij} (distance, time or cost).
  - Objective is to minimize total length of the route, total travel time or total travel cost.

Transportation and Routing Problems

Škoda Auto University

Treveling Salesman Problem – mathematical model

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if a vehicle travels directly between nodes } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

$$u_i = \text{ dummy variable in sub-tours eliminating constraints}$$

- **Model**

$$z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} \rightarrow \min,$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1,2,\ldots,n, \qquad\qquad \sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1,2,\ldots,n,$$

$$u_i + 1 - (n-1)(1 - x_{ij}) \le u_j \qquad i = 1,2,\ldots,n;\ \ j = 2,3,\ldots,n,$$

$$x_{ij} \in \mathbf{B} \qquad i = 1,2,\ldots,n;\ \ j = 1,2,\ldots,n,$$

$$u_i \in \mathbf{R}_+ \qquad i = 1,2,\ldots,m.$$

```
TITLE TSP;

OPTIONS
EXCELWORKBOOK="OTSP.xlsm";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("city");
j:=i;

DATA
c[i,j]:=EXCELRANGE("distance");
n:=count(i)-1;

BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("route");

VARIABLES
u[i] EXPORT TO EXCELRANGE("order");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
fromCity[i]:            sum(j:x[i,j])=1;
toCity[j]:              sum(i:x[i,j])=1;
subtours[i,j>1]:        u[i]+1-n*(1-x[i,j])<=u[j];

END
```

Transportation and Routing Problems

Škoda Auto University

Treveling Salesman Problem

- **Example**
  - A sales representative of the brewery located in Velvary must visit 7 pubs in 7 cities.
  - In the following table, distances (in km) correspond to direct links (roads) between cities. A dash indicates there is no direct road between cities.
  - The objective is to visit all pubs minimizing total length of the tour.

Town	Velvary	Kralupy	Libcice	Slaný	Zlonice	Vraný	Bříza	Veltrusy
Velvary	0	8	-	13	10	-	12	9
Kralupy	8	0	6	16	-	-	-	4
Libcice	-	6	0	-	-	-	-	-
Slaný	13	16	-	0	7	-	-	-
Zlonice	10	-	-	7	0	7	13	-
Vraný	-	-	-	-	7	0	15	-
Bříza	12	-	-	-	13	15	0	13
Veltrusy	9	4	-	-	-	-	13	0

Transportation and Routing Problems

Škoda Auto University

Treveling Salesman Problem

- **Example**
  - The following table contains distances between all pairs of cities.

Town	Velvary	Kralupy	Libcice	Slaný	Zlonice	Vraný	Bříza	Veltrusy
Velvary	0	8	14	13	10	17	12	9
Kralupy	8	0	6	16	18	25	17	4
Libcice	14	6	0	22	24	31	23	10
Slaný	13	16	22	0	7	14	20	20
Zlonice	10	18	24	7	0	7	13	19
Vraný	17	25	31	14	7	0	15	26
Bříza	12	17	23	20	13	15	0	13
Veltrusy	9	4	10	20	19	26	13	0

Transportation and Routing Problems

Treveling Salesman Problem – feasible solution

- **The Nearest Neighbor Algorithm**
  1. Select any node as the initial one of the tour.
  2. Find the nearest node (not selected before) to the last node and add it to the tour. If it is impossible (all nodes have been selected) then add the initial node to the tour (Hamiltonian cycle is created) and go to Step 4.
  3. Go to Step 2.
  4. End.

- Objective value

$$z = 85$$

Town	Velvary	Kralupy	Libcice	Slaný	Zlonice	Vraný	Bříza	Veltrusy
Velvary	0	8	14	13	10	17	12	9
Kralupy	8	0	6	16	18	25	17	4
Libcice	14	6	0	22	24	31	23	10
Slaný	13	16	22	0	7	14	20	20
Zlonice	10	18	24	7	0	7	13	19
Vraný	17	25	31	14	7	0	15	26
Bříza	12	17	23	20	13	15	0	13
Veltrusy	9	4	10	20	19	26	13	0

Transportation and Routing Problems

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Treveling Salesman Problem – optimal solution

- **Optimal solution**
  - Objective value

$$z_0 = 79$$

Town	Velvary	Kralupy	Libcice	Slaný	Zlonice	Vraný	Bříza	Veltrusy
Velvary	0	8	14	13	10	17	12	9
Kralupy	8	0	6	16	18	25	17	4
Libcice	14	6	0	22	24	31	23	10
Slaný	13	16	22	0	7	14	20	20
Zlonice	10	18	24	7	0	7	13	19
Vraný	17	25	31	14	7	0	15	26
Bříza	12	17	23	20	13	15	0	13
Veltrusy	9	4	10	20	19	26	13	0

Transportation and Routing Problems

Škoda Auto University

Treveling Salesman Problem with Time Windows

- **Definition of the problem**
  - Each location i has to be visited within time interval $<e_i, l_i>$.
  - A vehicle spends given time S_i at location i.
  - Let d_{ij} be the travel time between locations i and j.
  - The objective is to determine the minimal Hamiltonian cycle (in terms of distance) respecting all time windows.

Transportation and Routing Problems

Treveling Salesman Problem with Time Windows – mathematical model

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if a vehicle travels directly between locations } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

$$t_i = \text{ time of vehicle arrival to location } i$$

- **Model**
  - Variables u_i are eliminated and constraints are replaced with

$$t_i + S_i + d_{ij} - M(1 - x_{ij}) \le t_j \qquad i = 1, 2, \ldots, n; \ \ j = 2, 3, \ldots, n; \ \ i \ne j,$$

$$e_i \le t_i \le l_i \qquad i = 2, 3, \ldots, n,$$

$$t_1 = 0,$$

$$t_i \in \mathbf{R}_+ \qquad i = 2, 3, \ldots, n.$$

```
TITLE TSPTW;

OPTIONS
EXCELWORKBOOK="TSPTW.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("city");
j:=i;

DATA
c[i,j]:=EXCELRANGE("distance");
e[i]:=EXCELRANGE("earliest");
l[i]:=EXCELRANGE("latest");
S[i]:=EXCELRANGE("stay");
d[i,j]:=EXCELRANGE("time");
M:=1000;

BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("route");

VARIABLES
t[i] EXPORT TO EXCELRANGE("arrival");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
fromCity[i]:      sum(j:x[i,j])=1;
toCity[j]:        sum(i:x[i,j])=1;
travel[i,j>1]:    t[i]+S[i]+d[i,j]-M*(1-x[i,j])<=t[j];
window[i>1]:      e[i]<=t[i]<=l[i];
startingPoint:    t[1]=0;

END
```

Transportation and Routing Problems

Treveling Salesman Problem with Time Windows – optimal solution

- **Optimal solution**
  - Objective value

$$z_0 = 100$$

21	Route	Velv	Kra	Lib	Sla	Zlo	Vra	Bri	Velt	Arrival
22	Velvary	0	0	0	0	0	0	0	1	0
23	Kralupy	1	0	0	0	0	0	0	0	360
24	Libcice	0	0	0	1	0	0	0	0	99
25	Slany	0	0	0	0	0	1	0	0	146
26	Zlonice	0	0	0	0	0	0	1	0	268
27	Vrany	0	0	0	0	1	0	0	0	240
28	Briza	0	1	0	0	0	0	0	0	303
29	Veltrusy	0	0	1	0	0	0	0	0	60
30										
31	Total	100								

Route	Earliest	Latest	Arrival
Velvary	8:00	-	-
Veltrusy	9:00	11:00	9:00
Libcice	8:30	11:00	9:39
Slany	9:00	14:00	10:26
Vrany	9:00	12:00	12:00
Zlonice	11:30	15:00	12:28
Briza	13:00	15:30	13:03
Kralupy	14:00	16:00	14:00
Velvary	-	-	14:35

Transportation and Routing Problems

Škoda Auto University

Open Treveling Salesman Problem

- **Definition of the problem**
  - Set of customers.
  - Each customer must be visited exactly once.
  - Vehicle does not return to the home city (index 1).
  - Objective is to minimize total length of the route, total travel time or total travel cost.

Transportation and Routing Problems

Škoda Auto University

Open Treveling Salesman Problem – optimal solution

- **Optimal solution**
  - Objective value

$$z_0 = 65$$

	A	B	C	D	E	F	G	H	I	J
1	Distance	Velv	Kra	Lib	Sla	Zlo	Vra	Bri	Velt	
2	Velvary	0	8	14	13	10	17	12	9	
3	Kralupy	0	0	6	16	18	25	17	4	
4	Libcice	0	6	0	22	24	31	23	10	
5	Slany	0	16	22	0	7	14	20	20	
6	Zlonice	0	18	24	7	0	7	13	19	
7	Vrany	0	25	31	14	7	0	15	26	
8	Briza	0	17	23	20	13	15	0	13	
9	Veltrusy	0	4	10	20	19	26	13	0	
10										
11	Route	Velv	Kra	Lib	Sla	Zlo	Vra	Bri	Velt	Order
12	Velvary	0	0	0	1	0	0	0	0	0
13	Kralupy	0	0	1	0	0	0	0	0	6
14	Libcice	1	0	0	0	0	0	0	0	7
15	Slany	0	0	0	0	1	0	0	0	1
16	Zlonice	0	0	0	0	0	1	0	0	2
17	Vrany	0	0	0	0	0	0	1	0	3
18	Briza	0	0	0	0	0	0	0	1	4
19	Veltrusy	0	1	0	0	0	0	0	0	5
20										
21	Total	65								

Transportation and Routing Problems

Treveling Salesman Problem in Production

- **Definition of the problem**
  - The company produces utility glass (beakers, ashtrays, bottles, bowls, vases and test tubes). The production line measures the size of the drop of glass from the storage room into the press, on which the mold for the given product is placed. The conversion times (in hours) for changing the mold and setting the drop size are given in the table.
  - The company wants to minimize costs that correspond with downtime due to changing the mold and setting the drop.

a) The production batch always begins and ends with the production of a beaker.

b) The production batch begins with the production of the beaker and ends with any product.

c) The production batch begins with the production of the beaker and ends with the production of the bottle.

Product	Beaker	Ashtray	Bottle	Bowl	Vase	Test tube
Beaker	0	6	8	11	5	7
Ashtray	4	0	2	7	4	7
Bottle	5	12	0	10	7	6
Bowl	4	7	5	0	7	10
Vase	7	11	10	7	0	6
Test tube	10	7	15	13	15	0

Transportation and Routing Problems

Škoda Auto University

Treveling Salesman Problem in Production – optimal solution

a) The production batch always begins and ends with the production of a beaker.

Product	Beaker	Ashtray	Bottle	Bowl	Vase	Test tube
Beaker	0	6	8	11	5	7
Ashtray	4	0	2	7	4	7
Bottle	5	12	0	10	7	6
Bowl	4	7	5	0	7	10
Vase	7	11	10	7	0	6
Test tube	10	7	15	13	15	0

Product	Beaker	Ashtray	Bottle	Bowl	Vase	Test tube	Order
Beaker	0	0	0	0	0	1	0.
Ashtray	0	0	0	0	1	0	2.
Bottle	1	0	0	0	0	0	5.
Bowl	0	0	1	0	0	0	4.
Vase	0	0	0	1	0	0	3.
Test tube	0	1	0	0	0	0	1.

- Objective value $z_0 = 35$

```
TITLE Glass1;

OPTIONS
EXCELWORKBOOK="Glass.xlsx";
EXCELSHEETNAME="MPL1";

INDEX
i:=EXCELRANGE("product");
j:=i;

DATA
c[i,j]:=EXCELRANGE("time");
n:=count(i)-1;

BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("schedule");

VARIABLES
u[i] EXPORT TO EXCELRANGE("order");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
fromProduct[i]:        sum(j:x[i,j])=1;
toProduct[j]:          sum(i:x[i,j])=1;
subtours[i,j>1]:       u[i]+1-n*(1-x[i,j])<=u[j];

END
```

Transportation and Routing Problems

Škoda Auto University

Treveling Salesman Problem in Production – optimal solution

b) The production batch begins with the production of the beaker and ends with any product.

Product	Beaker	Ashtray	Bottle	Bowl	Vase	Test tube
Beaker	0	6	8	11	5	7
Ashtray	0	0	2	7	4	7
Bottle	0	12	0	10	7	6
Bowl	0	7	5	0	7	10
Vase	0	11	10	7	0	6
Test tube	0	7	15	13	15	0

Product	Beaker	Ashtray	Bottle	Bowl	Vase	Test tube	Order
Beaker	0	0	0	0	1	0	0
Ashtray	0	0	1	0	0	0	3
Bottle	0	0	0	0	0	1	4
Bowl	0	1	0	0	0	0	2
Vase	0	0	0	1	0	0	1
Test tube	1	0	0	0	0	0	5

- Objective value $z_0 = 27$

```
TITLE Glass2;

OPTIONS
EXCELWORKBOOK="Glass.xlsx";
EXCELSHEETNAME="MPL2";

INDEX
i:=EXCELRANGE("product");
j:=i;
v[i]:=EXCELRANGE("v");

DATA
c[i,j]:=EXCELRANGE("time");
n:=count(i)-1;

BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("schedule");

VARIABLES
u[i] EXPORT TO EXCELRANGE("order");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
fromProduct[i]:                        sum(j:x[i,j])=1;
toProduct[j]:                          sum(i:x[i,j])=1;
subtours[i,j] WHERE (NOT (j IN v)):  u[i]+1-n*(1-x[i,j])<=u[j];

END
```

Transportation and Routing Problems

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Treveling Salesman Problem in Production – solution

c) The production batch begins with the production of the beaker and ends with the production of the bottle.

Product	Beaker	Ashtray	Bottle	Bowl	Vase	Test tube
Beaker	0	6	8	11	5	7
Ashtray	0	0	2	7	4	7
Bottle	0	12	0	10	7	6
Bowl	0	7	5	0	7	10
Vase	0	11	10	7	0	6
Test tube	0	7	15	13	15	0

Product	Beaker	Ashtray	Bottle	Bowl	Vase	Test tube	Order
Beaker	0	0	0	0	1	0	0.
Ashtray	0	0	0	1	0	0	3.
Bottle	1	0	0	0	0	0	5.
Bowl	0	0	1	0	0	0	4.
Vase	0	0	0	0	0	1	1.
Test tube	0	1	0	0	0	0	2.

- Objective value $z_0 = 30$

```
TITLE Glass3;

OPTIONS
EXCELWORKBOOK="Glass.xlsx";
EXCELSHEETNAME="MPL3";

INDEX
i:=EXCELRANGE("product");
j:=i;

DATA
c[i,j]:=EXCELRANGE("time");
n:=count(i)-1;

BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("schedule");

VARIABLES
u[i] EXPORT TO EXCELRANGE("order");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
fromProduct[i]:   sum(j:x[i,j])=1;
toProduct[j]:     sum(i:x[i,j])=1;
subtours[i,j>1]:  u[i]+1-n*(1-x[i,j])<=u[j];
x[3,1]=1;

END
```

Transportation and Routing Problems

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Vehicle Routing Problem

- **Definition of the problem**
  - $G = \{V, E\}$ be a complete digraph with distance c_{ij} given for each arc (i, j).
  - Let node 1 be a depot, where one vehicle with capacity V is available, $|U| = n$.
  - Each customer i associated with request of size q_i.
  - The objective is to satisfy all customers' requirements and to minimize total length of the routes.
- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if a vehicle travels directly between nodes } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

$$u_i = \text{ dummy variable for the balance of load on the vehicle}$$

- **Assumptions**

$$\sum_{i=2}^{n} q_i > V,$$

$$q_i \leq V \qquad i = 2, 3, \ldots, n.$$

Transportation and Routing Problems

Vehicle Routing Problem – mathematical model

- **Model**

$$z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij} \rightarrow \min,$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 2,3,\ldots,n,$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 2,3,\ldots,n,$$

$$u_i + q_j - V(1 - x_{ij}) \le u_j \qquad i = 1,2,\ldots,n;\ j = 2,3,\ldots,n,$$

$$u_i \le V \qquad i = 2,3,\ldots,n,$$

$$u_1 = 0,$$

$$x_{ij} \in \mathbf{B} \qquad i = 1,2,\ldots,n;\ j = 1,2,\ldots,n,$$

$$u_i \in \mathbf{R}_+ \qquad i = 2,3,\ldots,n.$$

```
TITLE VRP;

OPTIONS
EXCELWORKBOOK="VRP.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("city");
j:=i;

DATA
c[i,j]:=EXCELRANGE("distance");
q[i]:=EXCELRANGE("requirement");
V:=EXCELRANGE("capacity");


BINARY VARIABLES
x[i,j] EXPORT TO EXCELRANGE("routes");

VARIABLES
u[i] EXPORT TO EXCELRANGE("load");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
fromCity[i>1]:          sum(j:x[i,j])=1;
toCity[j>1]:            sum(i:x[i,j])=1;
loadBalance[i,j>1]:     u[i]+q[j]-V*(1-x[i,j])<=u[j];
capacity[i>1]:          u[i]<V;
startingPoint:          u[1]=0;

END
```

Transportation and Routing Problems

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Vehicle Routing Problem

- **Example**
  - The sales representative of the brewery has arranged advantageous contracts.
  - Pubs will take barrels of beer in quantities given in the following table. For delivery, a vehicle with the capacity of 50 barrels will be used.
  - The objective is to satisfy all requirements minimizing total length of the vehicle tours.

	Requirement
Velvary	0
Kralupy	18
Libcice	10
Slany	15
Zlonice	12
Vrany	10
Briza	8
Veltrusy	11

Transportation and Routing Problems

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Vehicle Routing Problem – optimal solution

- **Optimal solution**
  - Objective value

$$z_0 = 87$$

Route1	Requirement
Velvary	0
Kralupy	18
Libcice	10
Veltrusy	11
Velvary	0
Celkem	39

Route1	Requirement
Velvary	0
Briza	8
Vrany	10
Zlonice	12
Slany	15
Velvary	0
Celkem	45

Transportation and Routing Problems

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Pickup and Delivery Problems (PDP)

- **One-to-One PDP**
  - For each requirement a pick-up location and a delivery location are given. Vehicle routes start and end at a common depot.
  - Transport of handicapped persons (Dial-a-Ride Problem), Messenger Problem etc.
- **Many-to-Many PDP**
  - Commodity may be picked up at one of many locations and also be delivered to one of many locations.
  - For example, shared bicycles or scooters.
- **One-to-Many-to-One PDP**
  - Each customer receives a delivery originating at a common depot and sends picked up quantity to the depot.
  - Barrels transport between brewery and restaurants etc.

Transportation and Routing Problems

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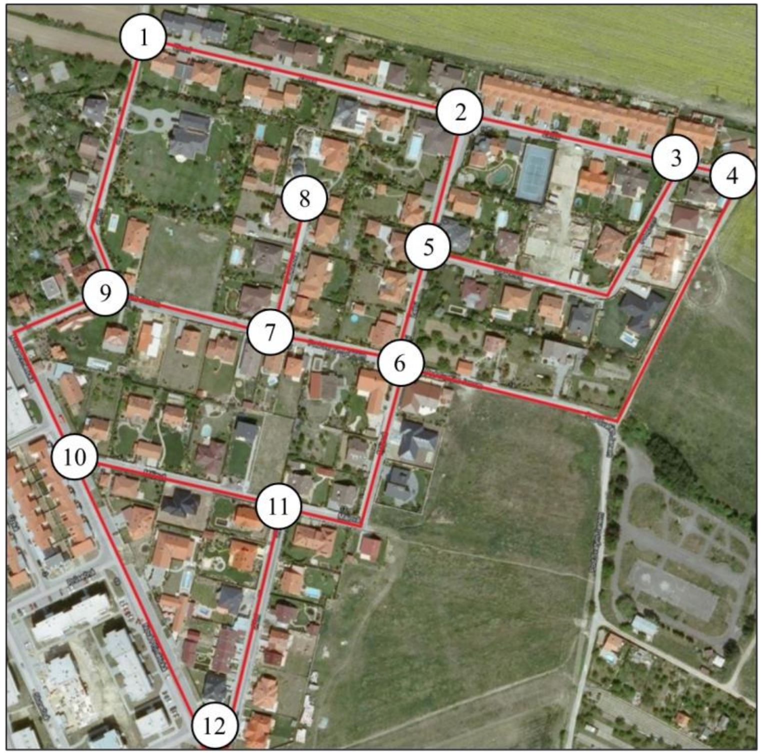
Undirected Chinese Postman Problem

- **Example**
  - At Halloween, trik-or-treating children want to visit all houses in neighborhood. The lengths of streets (in meters), they must go through, are given in the table.
  - Will you plan a tour for children to minimize the total distance.

Arc	Length	Arc	Length
(1,2)	210	(6,7)	80
(1,9)	160	(6,11)	150
(2,3)	140	(7,8)	80
(2,5)	80	(7,9)	110
(3,4)	40	(9,10)	160
(3,5)	210	(10,11)	130
(4,6)	310	(10,12)	190
(5,6)	70	(11,12)	150



Transportation and Routing Problems

Undirected Chinese Postman Problem – mathematical model

- **Decision variables**

  $x_{ij} =$ a number of copies of arc (i, j) in supergraph G^*

- **Model**

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \rightarrow \min,$$

$$x_{ij} + x_{ji} \geq r_{ij} \qquad i, j = 1, 2, \ldots, n,$$

$$\sum_{j=1}^{n} x_{ji} = \sum_{j=1}^{n} x_{ij} \qquad i = 1, 2, \ldots, n,$$

$$x_{ij} \in \mathbf{Z}_+ \qquad i, j = 1, 2, \ldots, n.$$

$$r_{ij} = \begin{cases} 1 & \text{jestliže existuje hrana } (i, j) \\ 0 & \text{jinak} \end{cases}$$

```
TITLE UCPP;

OPTIONS
EXCELWORKBOOK="UCPP.xlsx";
EXCELSHEETNAME="MPL";

INDEX
i:=EXCELRANGE("node");
j:=i;

DATA
c[i,j]:=EXCELRANGE("length");
r[i,j]:=EXCELRANGE("arc");

INTEGER VARIABLES
x[i,j] EXPORT TO EXCELRANGE("number");

MODEL
MIN z EXPORT TO EXCELRANGE("total") =sum(c*x);

SUBJECT TO
arc[i,j]:      x[i,j]+x[i:=j,j:=i]>=r[i,j];
Euler[i]:      SUM(j:x[i,j])=SUM(j:x[i:=j,j:=i]);

END
```

Transportation and Routing Problems

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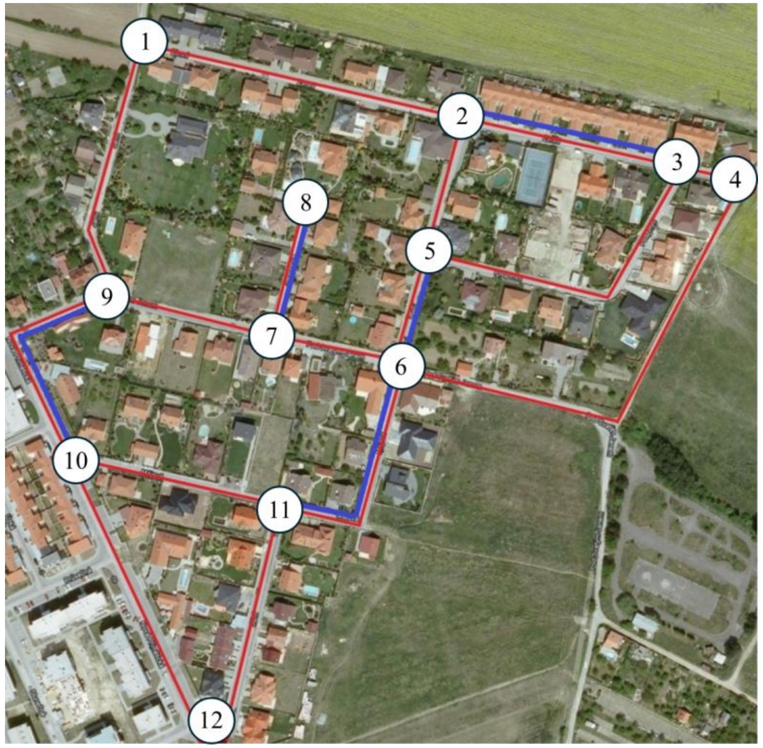
Undirected Chinese Postman Problem – optimal solution

- **Optimal solution**
  - Objective value

$$z_0 = 2870$$

31	Number	1	2	3	4	5	6	7	8	9	10	11	12
32	1	0	1	0	0	0	0	0	0	0	0	0	0
33	2	0	0	1	0	1	0	0	0	0	0	0	0
34	3	0	1	0	0	1	0	0	0	0	0	0	0
35	4	0	0	1	0	0	0	0	0	0	0	0	0
36	5	0	0	0	0	0	2	0	0	0	0	0	0
37	6	0	0	0	1	0	0	1	0	0	0	1	0
38	7	0	0	0	0	0	0	0	1	1	0	0	0
39	8	0	0	0	0	0	0	1	0	0	0	0	0
40	9	1	0	0	0	0	0	0	0	0	1	0	0
41	10	0	0	0	0	0	0	0	0	1	0	1	0
42	11	0	0	0	0	0	1	0	0	0	0	0	1
43	12	0	0	0	0	0	0	0	0	0	1	0	0
44													
45	Total	2870											



Route: 1, 2, 3, 4, 6, 11, 12, 10, 9, 10, 11, 6, 5, 3, 2, 5, 6, 7, 8, 7, 9, 1.

Transportation and Routing Problems

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Other Variations of Chinese Postman Problem

- **Directed Chinese Postman Problem** (municipal waste collection in one-way streets)
- **Mixed Chinese Postman Problem** (street clearing and sprinkling)
- **Capacitated Postman Problem** (municipal waste collection)
- **Rural Postman Problem** (mail delivery)
- **Windy Postman Problem** (mail delivery)
- **Hierarchical Postman Problem** (snow plowing)

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Thank you for attention

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