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Operational Research I

Lectures

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Literature

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Basic

- FÁBRY, J. Operational Research I for full-time and distance form of studies. Mladá Boleslav: ŠAU, 2022. 150 pp. ISBN 978-80-7654-048-4.
- EISELT, H. and SANDBLOM, C. *Operations Research.: A Model-Based Approach*. 1st edition, Heidelberg: Springer, 2010. 446 pp. ISBN 978-3-642-10325-4.
- Fábry, J. *Management Science*. University of Economics Prague, 2003. ISBN 80-245-0586–X (Available at https://janfabry.cz/Management-Science.pdf).

Recommended

- HILLIER, F. S. and LIEBERMAN, G. J. *Introduction to Operations Research*. 11th edition, McGraw-Hill, 2021. 964 pp. ISBN 9781260575873
- BOUCHERIE, R. J., BRAAKSMA, A. and TIJMS, H. *Operations Research: Introduction to Models and Methods*. World Scientific, 2022. 499 pp. ISBN 9789811239342
- RARDIN, R. L. *Optimization in Operations Research*. 2nd edition, Pearson, 2018. 1144 pp. ISBN 978-93-530-6636-9

Contents

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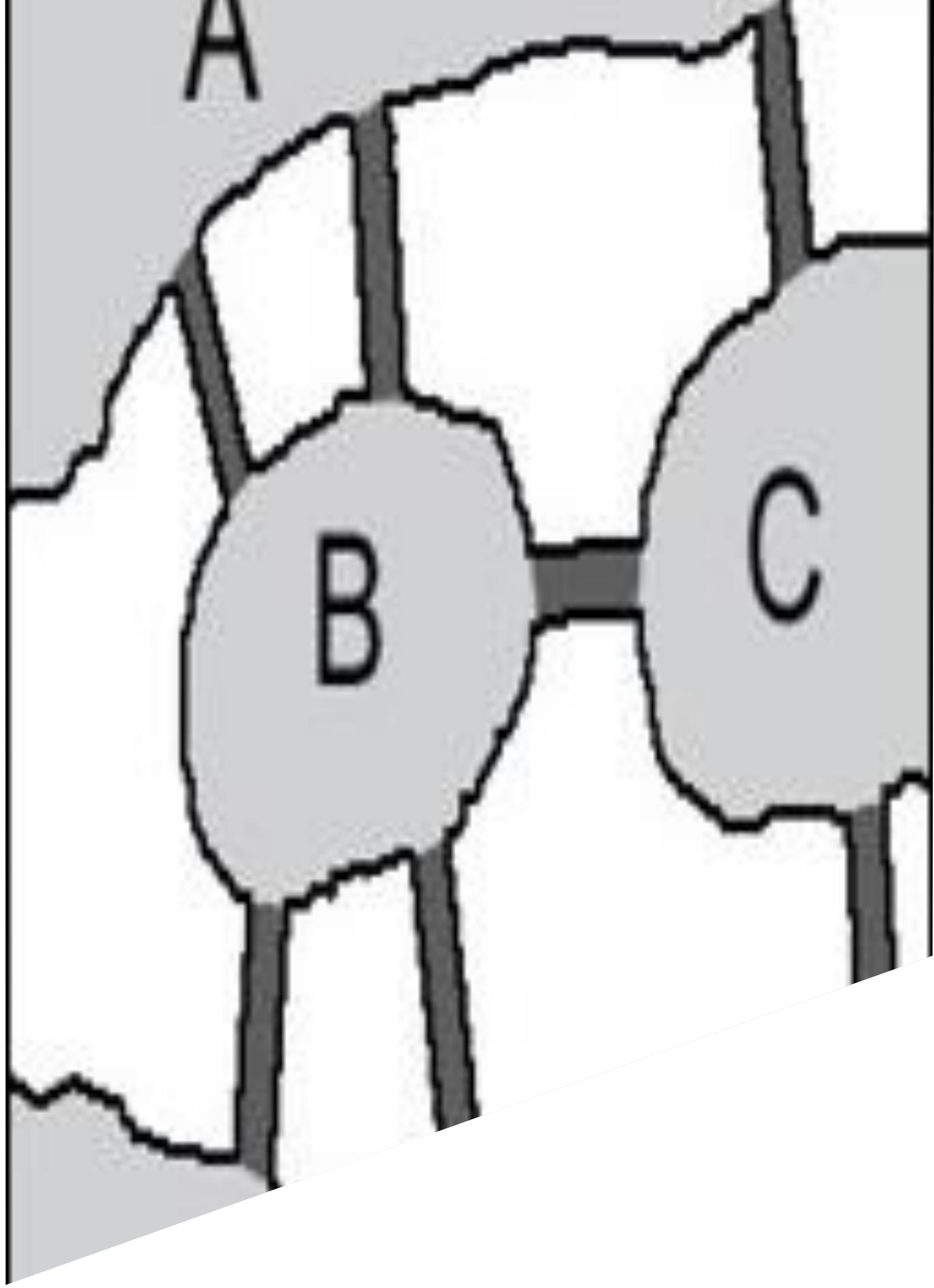
Introduction to Operational Research



INTERNAL

Introduction to Operational Research

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Alternative Names and Related Fields

- Operational / Operations Research (OR)
- Management Science (MS)
- Operations Analysis
- Quantitative Analysis
- Quantitative Methods
- Systems Analysis
- Decision Analysis
- Decision Science
- Computer Science

Introduction to Operational Research

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Definition

1. OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problems.
2. MS/OR is the application of the scientific method to the study of the operations of large, complex organizations or activities.
3. MS/OR is the application of the scientific method to the analysis and solution of managerial decision problems.

- **Summary**
  - Application of SCIENTIFIC METHOD.
  - Study of LARGE & COMPLEX SYSTEMS.
  - Analysis of MANAGERIAL PROBLEMS.
  - Finding OPTIMAL SOLUTION.
  - Use of MATHEMATICAL MODELS.
  - Use of COMPUTERS & SPECIAL SOFTWARE.

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Software

- MPL for Windows
- AMPL
- Lingo (LINDO)
- XPRESS (FICO)
- CPLEX (IBM ILOG)
- Gurobi
- AIMMS
- NEOS
- MS Excel (FRONTLINE SOLVERS)
- PLANT SIMULATION
- SIMPROCESS
- SIMUL 8
- Matlab

Introduction to Operational Research

Decision-making

- Two or more alternatives.
- Conclusion = Decision.
- Systematic process.

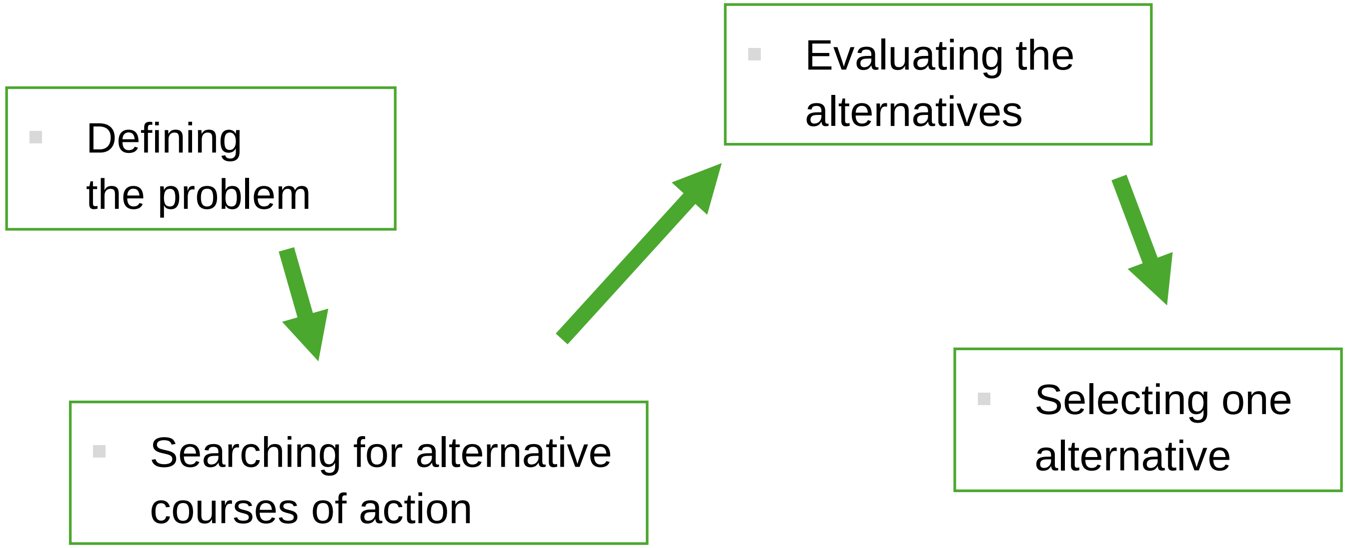

Fig. 1 – Scheme of Decision-making Process

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Modeling Process (Analytical Approach)

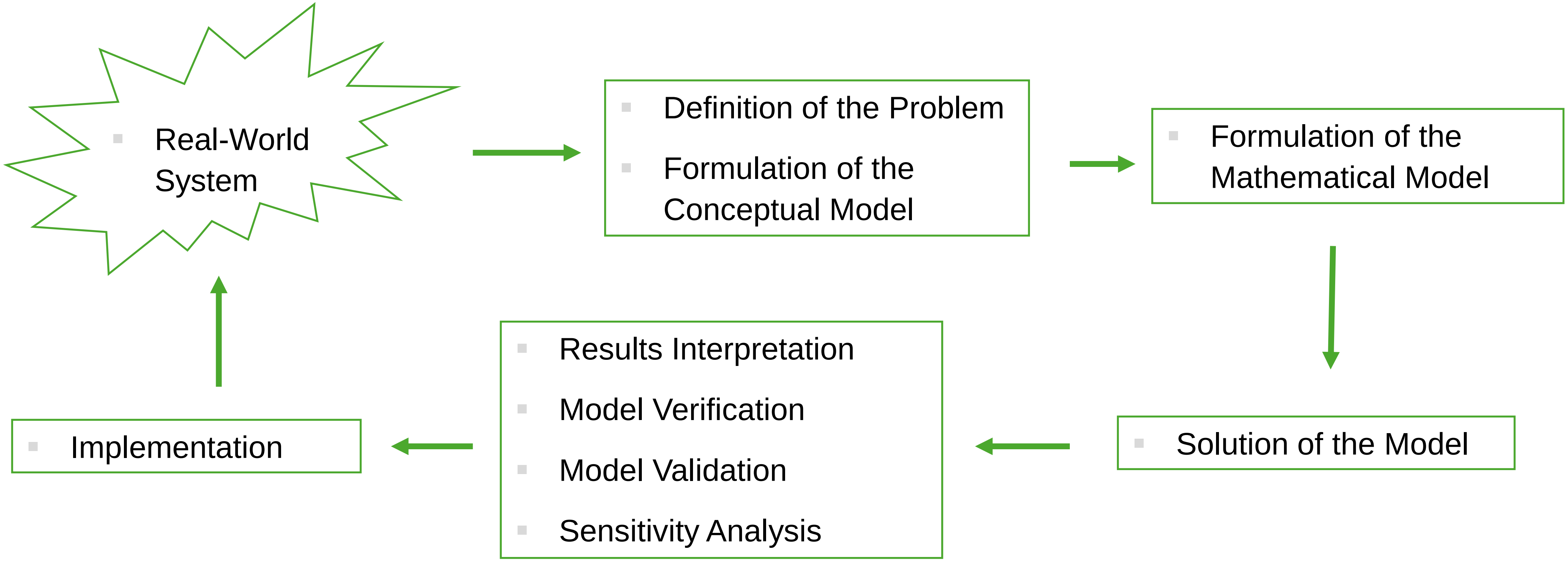

Fig. 2 – Scheme of Modeling Process

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Modeling

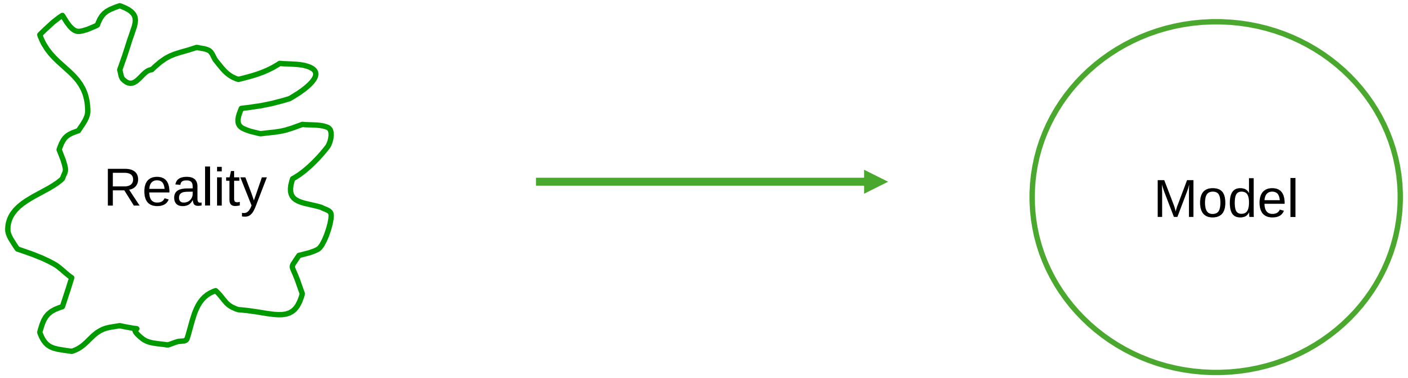


Fig. 3 – Simplification of Reality

- Finding a proper balance between the level of simplification of the model and the good representation of reality.

Introduction to Operational Research

Models

- Deterministic – all parameters are known with certainty.
- Probabilistic (stochastic) – some parameters are values of random variables.

- Static – all data is known in advance (before solution process).
- Dynamic – data can be changed after solution is obtained.

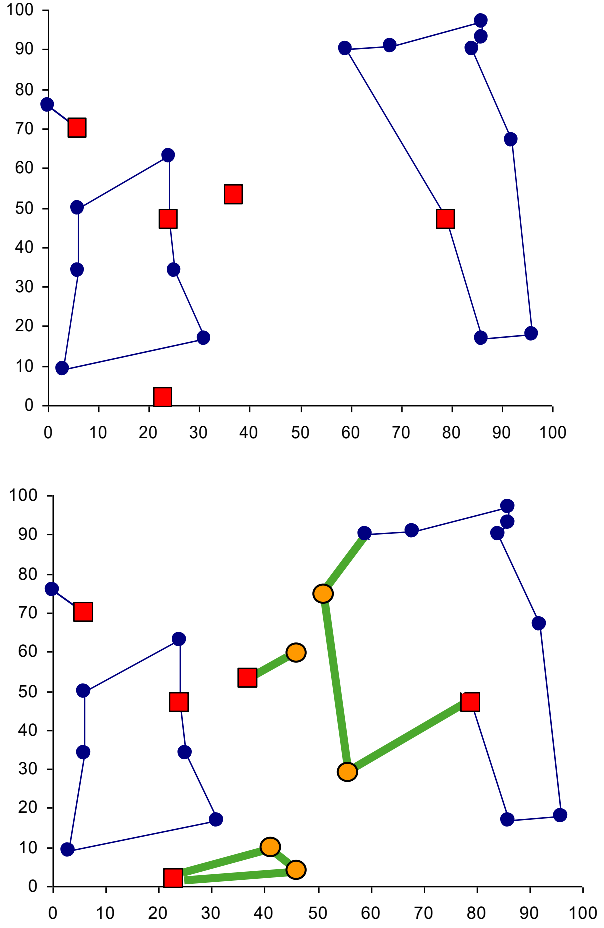
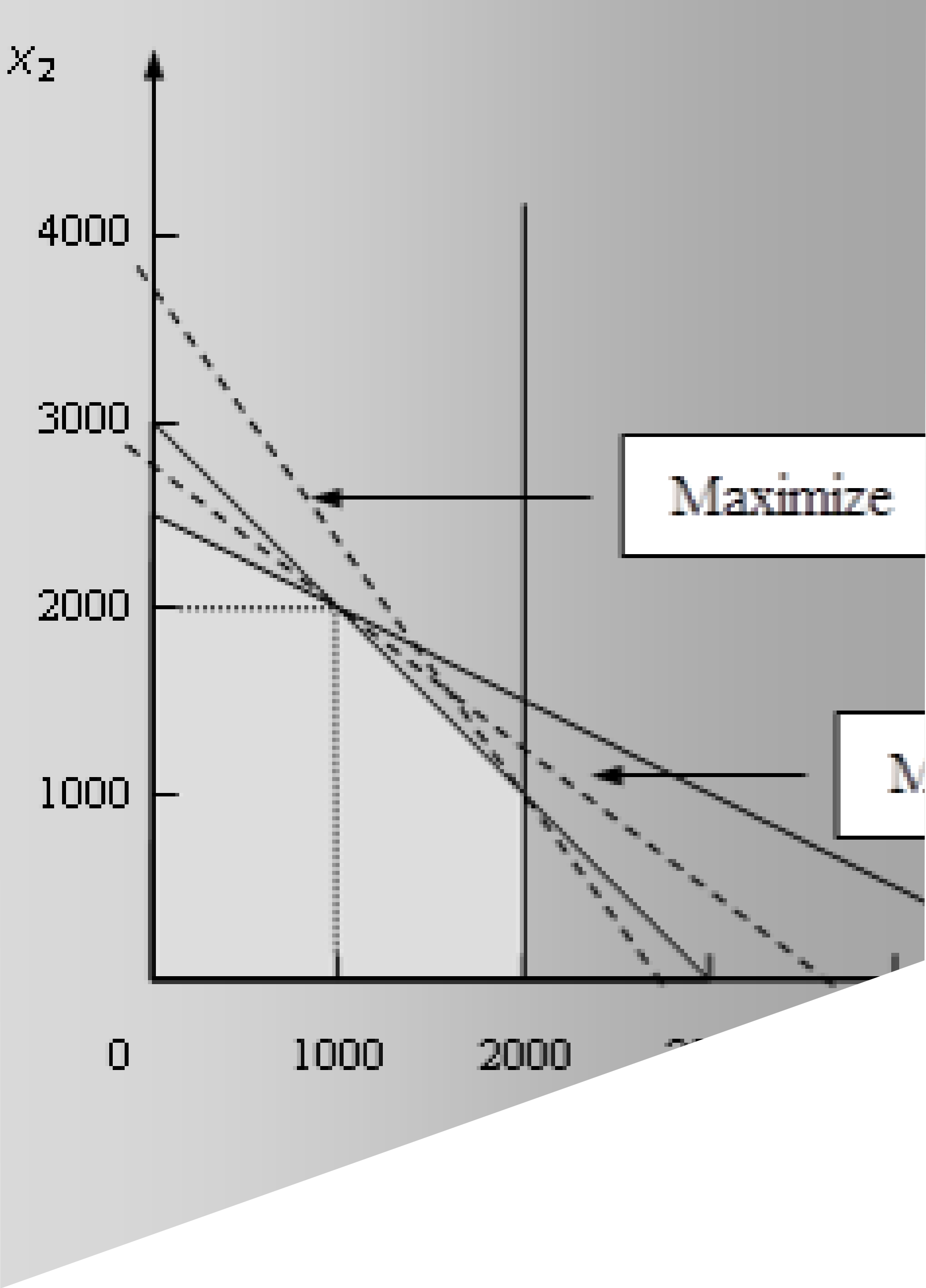


Fig. 4 – Static and Dynamic Model

2

Linear Programming



INTERNAL

Linear Programming

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Introduction

- **Conceptual model**
  - Processes.
  - Restrictions.
  - Objective.
- **Mathematical model**
  - Decision Variables – continuous, integral, binary.
  - Constraints – equations, inequalities.
  - Objective function – max, min.
- **Solution**
  - Feasible – satisfies all constraints.
  - Optimal – best feasible solution in terms of the objective.
  - Infeasible – does not satisfy any constraint.

Linear Programming

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Introduction

- **Solution**
  - Results Interpretation – explanation of values to the others (e.g. client).
  - Model Verification – comparison of the mathematical model with the conceptual model.
  - Model Validation – comparison of results with the real expectations.
  - Sensitivity Analysis – examination of the impact of changes in inputs on outputs.
- **Implementation**
  - Use of results in real system.
- **Special situations of LP problems**
  - Unique optimal solution.
  - Multiple optimal solution.
  - No optimal solution.
  - No feasible solution.

Linear Programming

Production Planning Problem

- **Example**
  - The company manufactures 2 types of wooden toys: trucks and trains.
  - The price of a piece of truck is 820 CZK, of a piece of train 1150 CZK. The wood cost for the truck is 100 CZK, whereas for the train 180 CZK.
  - The truck requires 1 hour of carpentry labor and 1 hour of finishing labor (assembling and painting). The train requires 2 hours of carpentry labor and 1 hour of finishing labor.
  - Worth of carpentry labor is 150 CZK per hour, worth of finishing labor is 120 CZK per hour.
  - Each month, the company has 5000 available hours of carpentry labor and 3000 hours of finishing labor.
  - Demand for trains is unlimited, but at most 2000 trucks are, at an average, bought each month.
  - The Pinocchio’s management wants to maximize monthly profit (total revenue - total cost).

Linear Programming

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Production Planning Problem

- **Decision variables**

  x_1 = number of trucks produced each month,

  x_2 = number of trains produced each month.

- **Mathematical model**

Maximize $z = 450x_1 + 550x_2$,

subject to

$x_1 + 2x_2 \le 5000$, (carpentry labor)

$x_1 + x_2 \le 3000$, (finishing labor)

$x_1 \le 2000$, (demand)

$x_1, x_2 \ge 0$, integers.

Slack variables →

Equivalent set of equations

$$x_1 + 2x_2 + x_3 = 5000$$
$$x_1 + x_2 + x_4 = 3000$$
$$x_1 + x_5 = 2000$$

$\le$ + Slack variable

$\ge$ – Surplus variable

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Production Planning Problem

- **Optimal solution**
  - Decision variables

$$x_1 = 1000$$
$$x_2 = 2000$$

  - Objective value

$$z_0 = 1550000$$

  - Slack/surplus variables

$$x_3 = 0$$
$$x_4 = 0$$
$$x_5 = 1000$$

Linear Programming

Blending Problem

- **Input**
  - chemicals,
  - metal alloys,
  - crude oil,
  - livestock feeds,
  - foodstuffs.
- **Output requirements and/or objective**
  - quality,
  - quantity,
  - cost.
- **Decision variables**
  - amount of ingredients used in the final blend.

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Blending Problem

- **Example**
  - Design the optimal composition of nutritive mix that will contain at least 100 units of proteins, at least 300 units of starch and will weight at least 200 kg with minimum purchasing costs.
  - In following table, there are given contents of proteins and starch in 1kg of each nutritive feed and prices for 1 kg of feed.

Tab. 1 – Contents of proteins and starch in feeds, price of feeds

	Feed F_1	Feed F_2	Feed F_3	Feed F_4
Proteins (units)	0	3	1	2
Starch (units)	1	2	3	0
Price (CZK)	20	80	60	30

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Blending Problem

- **Decision variables**

x_i = amount of feed F_i in the final blend ($i = 1,2,3,4$)

- **Mathematical model**

Minimize $z = 20x_1 + 80x_2 + 60x_3 + 30x_4$,

subject to

$$
\begin{aligned}
3x_2 + x_3 + 2x_4 &\geq 100, \quad \text{(proteins)}\\
x_1 + 2x_2 + 3x_3 &\geq 300, \quad \text{(starch)}\\
x_1 + x_2 + x_3 + x_4 &\geq 200, \quad \text{(weight)}\\
x_i &\geq 0, i = 1,2,3,4.
\end{aligned}
$$

Linear Programming

Blending Problem

- **Optimal solution**
  - Decision variables

$$x_1 = 120$$
$$x_2 = 0$$
$$x_3 = 60$$
$$x_4 = 20$$

  - Objective value

$$z_0 = 6600$$

  - Slack/surplus variables

$$x_5 = 0$$
$$x_6 = 0$$
$$x_7 = 0$$

Linear Programming

Cutting Stock Problem

- **Input – raw product (dimension)**
  - pipes, tubes (1D).
  - roles of paper or textile (1D or 2D),
  - wooden sticks or laths (1D),
  - wooden boards (1D or 2D),
  - steel plates (2D),
  - boxes (3D) – 3D rectangular packing problem.
- **Output – final or semi-finished products**
- **Objective**
  - minimization of total loss,
  - minimization of a number of raw products being cut,
  - maximization of number of final/assembled products,
  - maximization of profit ensuing from sold final/assembled products.

Linear Programming

Cutting Stock Problem

- **Table of all cutting patterns**
  - It contains all possibilities of cutting raw products.
  - Each cutting pattern corresponds to the variable giving a number of the raw products being cut according to this pattern.

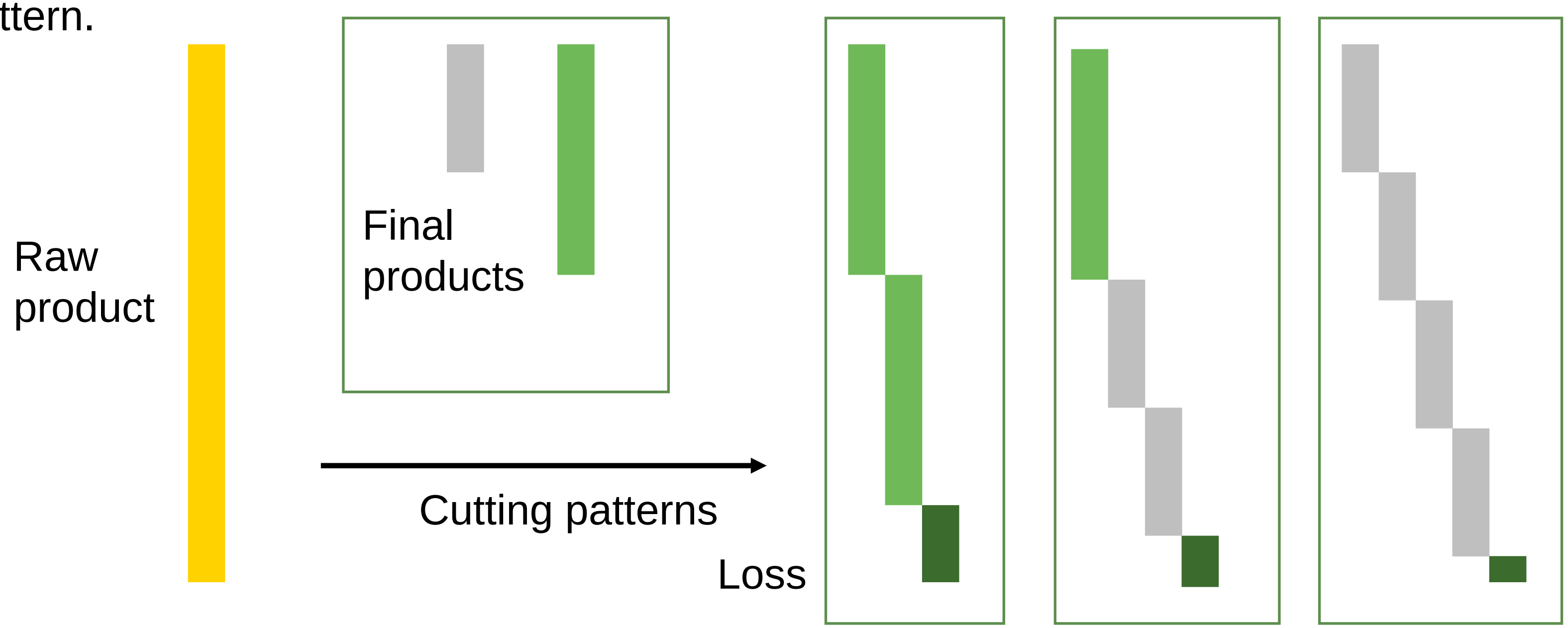

Fig. 5 – Cutting Patterns

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Cutting Stock Problem

- **Example**
  - The company produces bird feeders and bird houses. The producer decided to prepare a special collection for an exhibition (with possible sales) that should be held in 20 days.
  - The price of the bird feeder is set to 260 CZK, the price of the bird house is 570 CZK.
  - Material and time requirements for assembling both the products can be found in Table 2.

Tab. 2 – Material and time requirements

	Feeder	House
Boards 30 cm long	1	2
Boards 25 cm long	1	4
Screws	8	16
Time (in min)	30	60

Linear Programming

Cutting Stock Problem

- **Example**
  - There is available stock of raw boards: 500 boards of length 1.1 m and 150 boards of length 1.4 m. These boards must be cut into final boards of length 30 cm and 25 cm.
  - Available stock of screws is 3000 pieces.
  - The producer can work 8 hours per day and intends to maximize the total revenue ensuring from the sales (all production is supposed to be sold).

Tab. 3 – Table of cutting patterns

	1.1 m board				1.4 m board				
Pattern	1	2	3	4	5	6	7	8	9
30 cm board	3	2	1	0	4	3	2	1	0
25 cm board	0	2	3	4	0	2	3	4	5
Loss (cm)	20	0	5	10	20	0	5	10	15

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Cutting Stock Problem

- **Decision variables**

  x_i = number of 1.1 m boards being cut according to i – th pattern (i = 1,…,4)

  x_i = number of 1.4 m boards being cut according to i – th pattern (i = 5,…,9)

  x_{10} = number of assembled bird feeders

  x_{11} = number of assembled bird houses

- **Mathematical model**

Maximize $z = 260x_{10} + 570x_{11}$

subject to

$x_1 + x_2 + x_3 + x_4 \leq 500$ (1.1 m boards)

$x_5 + x_6 + x_7 + x_8 + x_9 \leq 150$ (1.4 m boards)

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Cutting Stock Problem

- **Mathematical model**

$$8x_{10} + 16x_{11} \leq 3000$$ (screws)

$$0.5x_{10} + x_{11} \leq 160$$ (time)

$$3x_1 + 2x_2 + x_3 + 4x_5 + 3x_6 + 2x_7 + x_8 \geq x_{10} + 2x_{11}$$ (30 cm boards)

$$2x_2 + 3x_3 + 4x_4 + 2x_6 + 3x_7 + 4x_8 + 5x_9 \geq x_{10} + 4x_{11}$$ (25 cm boards)

$$x_1, x_2, ..., x_{11} \geq 0$$

$x_1, x_2, ..., x_{11}$ are integers

Linear Programming

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Cutting Stock Problem

- **Optimal solution**
  - Decision variables

$$x_1 = x_3 = x_4 = x_6 = x_7 = x_8 = x_{10} = 0$$
$$x_2 = 65$$
$$x_5 = 48$$
$$x_9 = 102$$
$$x_{11} = 160$$

  - Objective value

$$z_0 = 91200$$

  - Slack/surplus variables

$$y_1 = 435 \qquad y_4 = 0$$
$$y_2 = 0 \qquad y_5 = 2$$
$$y_3 = 440 \qquad y_6 = 0$$

- These values can differ because of multiple optimal solution exists

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Portfolio Selection Problem

- **Definition of the problem**
  - Financial planning problem.
  - Allocation of available amount of money in several investment alternatives.
  - Financial risk.
  - The goal is to gain a certain amount of money (return).
  - The objective is to maximize the total return and minimize the total risk.
- **Alternative investments**
  - Shares, bonds etc.
- **Decision-makers**
  - Mutual fund, bank, pension fund, insurance company, individual investor.

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Portfolio Selection Problem

- **Example**
  - The management of an investment company is considering investing money in the shares of 4 beverage companies.
  - In order to avoid losses from the risk associated with investing in the private sector, the company's management has decided to invest part of its money in government bonds.
  - The total amount invested is 2 million CZK. Long-term financial market monitoring provides the percentages of annual expected return and risk indices for the stocks considered, shown in Table 4.

Tab. 4 – Stock evaluation

Stock	Return rate	Risk index
Bohemian Beer share	12 %	0.07
Moravian Wine share	9 %	0.09
Moravian Brandy share	15 %	0.05
Bohemian Milk share	7 %	0.03
Government bond	6 %	0.01

Linear Programming

Portfolio Selection Problem

- **Example**
  - The following rules were decided at the management meeting:
    1) No more than 200000 CZK might be invested in Bohemian Milk shares.
    2) Government bonds should cover at least 20 % of all investments.
    3) Because of diversification of portfolio, neither alcohol-drink company should receive more than 800000 CZK.
    4) Risk index of the final portfolio should be maximally 0.05.
  - Satisfying all the restrictions, the management intends to maximize the expected annual return of the portfolio.

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Portfolio Selection Problem

- **Decision variables**

  x_i = the amount of money invested in $i-\text{th}$ stock title $(i = 1, 2, \dots, 5)$

- **Mathematical model**
  - Maximize $z = 0.12x_1 + 0.09x_2 + 0.15x_3 + 0.07x_4 + 0.06x_5$

    Subject to

    $x_1 + x_2 + x_3 + x_4 + x_5 = 2\,000,$ (2 millions CZK)

    $x_4 \leq 200,$ (no more than 200000 CZK in Bohemia Milk shares)

    $x_5 \geq 400,$ (at least 20 % of all investments in government bonds)

    $x_1 \leq 800;\ x_2 \leq 800; x_3 \leq 800,$ (no more than 800000 CZK in alcohol-drink companies)

    $\frac{0.07x_1+0.09x_2+0.05x_3+0.03x_4+0.01x_5}{2\,000} \leq 0.05,$ (overall portfolio risk index is below 0.05)

    $0.07x_1 + 0.09x_2 + 0.05x_3 + 0.03x_4 + 0.01x_5 \leq 100,$

    $x_i \geq 0, \quad i = 1,2,\dots,5.$ (nonnegativity constraints)

Linear Programming

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Portfolio Selection Problem

- **Optimal solution**
  - Decision variables

$$x_1 = 800,$$

$$x_2 = 0,$$

$$x_3 = 800,$$

$$x_4 = 0,$$

$$x_5 = 400.$$

  - Objective value

$$z_0 = 240.$$

Linear Programming

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Transportation Problem

- **Definition of the problem**
  - Transport of homogeneous product.
  - Set of sources with limited supply.
  - Set of destinations with demand (requirement).
  - Unit shipping cost for all pairs of sources and destinations.
  - The goal is to satisfy all requirements without exceeding any supply.
  - The objective is to find shipments to minimize total shipping cost.
- **Type of the problem**
  - Balanced – total supply is equal to total demand.
  - Unbalanced – total supply is different from total demand, it is possible to make the problem balanced:
    - adding dummy destination,
    - finding additional source or adding dummy source (with the possibility of unsatisfied requirement).

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Transportation Problem

- **Example**
  - The international company operating in the Czech Republic is going to establish three subsidiaries producing chips. They should be located in following cities: Benešov, Jihlava and Tábor.
  - The main ingredient potatoes would be supplied from two warehouses in Humpolec and Pelhřimov.

Fig. 6 – Shipping from warehouses to subsidiaries

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Transportation Problem

- **Example**
  - The management of the corporation has estimated the weekly requirements of the companies. The warehouses' capacities are limited. Potatoes are transported once a week from suppliers to destinations by train and it is possible to evaluate a unit shipping cost per ton. All the values are given in Table 5.
  - The objective is to determine such shipments from warehouses to destinations that minimize the total shipping cost. This shipping schedule, of course, must satisfy requirement of each destination, and must not exceed supply of any warehouse.

Tab. 5 – Supplies, demands and unit shipping costs

	Benešov	Jihlava	Tábor	Supply
Humpolec	330	250	350	70
Pelhřimov	300	240	250	80
Demand	45	60	35	

Linear Programming

Transportation Problem

- **Feasible solution (North-West Corner Method)**
  1. Select the upper left-hand corner cell of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand, i.e., min(s1, d1).
  2. Adjust the supply and demand numbers in the respective rows and columns.
  3. If the demand for the first cell is satisfied, then move horizontally to the next cell in the second column.
  4. If the supply for the first row is exhausted, then move down to the first cell in the second row.
  5. If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.
  6. Continue the process until all supply and demand values are exhausted.

Tab. 6 – Feasible solution, North-West Corner Method

Shipments	Benešov	Jihlava	Tábor	Supply
Humpolec	45	25	-	70
Pelhřimov	-	35	35	80
Demand	45	60	35	

- Objective value

$$z = 38250$$

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Transportation Problem

- **Feasible solution (Minimal Cost Method)**
  1. Find a minimal cost for all possible shipments.
  2. Assign the shipment to the pair od source and destination with the minimal cost found in step 1. The value of the shipment is equal to the minimum of remaining supply and remaining demand for this pair.
  3. Decrease remaining supply and remaining demand, for the pair of the source and destination, by the shipment calculated in step 2.
  4. If there is some remaining demand go to step 1, otherwise, the feasible solution is found.

Tab. 7 – Feasible solution, Minimal Cost Method

Shipments	Benešov	Jihlava	Tábor	Supply
Humpolec	45	-	15	70
Pelhřimov	-	60	20	80
Demand	45	60	35	

- Objective value

$$z = 39500$$

Linear Programming

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Transportation Problem

- **Decision variables**

  x_{ij} = amount of potatoes (in tons) transported from source i to destination j ($i = 1,2; j=1,2,3$)

- **Mathematical model**

  Minimize $z = 330x_{11} + 250x_{12} + 350x_{13} + 300x_{21} + 240x_{22} + 250x_{23}$

  subject to

  $x_{11} + x_{12} + x_{13} \leq 70$ (Humpolec)

  $x_{21} + x_{22} + x_{23} \leq 80$ (Pelhřimov)

  $x_{11} + x_{21} = 45$ (Benešov)

  $x_{12} + x_{22} = 60$ (Jihlava)

  $x_{13} + x_{23} = 35$ (Tábor)

  $x_{ij} \geq 0, \quad i = 1,2; \; j = 1,2,3$

Linear Programming

Transportation Problem

- **Mathematical model**

$$z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} \rightarrow \min$$

$$\sum_{j=1}^{n} x_{ij} \le a_i \,, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \,, \quad j = 1, 2, \ldots, n$$

$$x_{ij} \ge 0 \,, \quad i = 1, 2, \ldots, m;\; j = 1, 2, \ldots, n$$

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Transportation Problem

- **Optimal solution**
  - Decision variables

Tab. 8 – Optimal solution

Shipments	Benešov	Jihlava	Tábor	Supply
Humpolec	-	60	-	70
Pelhřimov	45	-	35	80
Demand	45	60	35	

  - Objective value

$$z_0 = 37250$$

  - Slack/surplus variables

$$y_1 = 10$$
$$y_2 = 0$$

Linear Programming

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Container Transportation Problem

- Based on the transportation problem.
- Goods is transported in containers of the same capacity.
- The shipping costs are not associated with the transported units, but with the use of one container between the source and the destination.
- The objective is to determine the shipments between sources and destinations and the number of containers used for the transport.

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Container Transportation Problem

- **Example**
  - Based on the previous example, the company charges the shipping costs for renting one wagon between sources and destinations.
  - Wagons with capacity of 18 tons will be used for transport.
  - The objective is to determine how many tons of potatoes will be shipped between the individual locations, but in addition, to determine how many wagons will be used for this transport so that the total transportation costs are minimal.

Tab. 9 – Statement of container transportation problem

	Benešov	Jihlava	Tábor	Supply
Humpolec	4200	4800	5300	70
Pelhřimov	5100	3400	3700	80
Demand	45	60	35	

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Container Transportation Problem

- Mathematical model is based on the mathematical model of transportation problem.
- **Decision variables**

  x_{ij} = the shipment (in tones) from $i-$ th source $j-$ th destination.

  y_{ij} = number of containers used for transport.
- **Objective**

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} y_{ij} \to \min$$

- **Constraints**
  - It is necessary to add following constraints to the constraints of the transportation problem:

$$x_{ij} \leq K y_{ij}\,, \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$$

$$y_{ij} \geq 0,\ \text{integers}, \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$$

Linear Programming

Container Transportation Problem

- **Optimal solution**

Tab. 10 – Shipments

	Benešov	Jihlava	Tábor
Humpolec	45	18	-
Pelhřimov	-	42	35

Tab. 11 – Number of wagons

	Benešov	Jihlava	Tábor
Humpolec	3	1	-
Pelhřimov	-	3	2

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Assignment Problem

- **Definition of the problem**
  - Two sets of items.
  - Each item from the first set is to be assigned to exactly one item form the second set.
  - Each item from the second set is to be assigned to exactly one item form the first set.
  - The assignment of each pair of items is evaluated.
  - The objective is to maximize/minimize total value of assignment.
  - The assumption: sizes of both sets are equal (balanced problem).

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Assignment Problem

- **Example**
  - The company gets four commissions for building family houses in various parts of Prague (Michle, Prosek, Radlice, Troja).
  - In the first step the company must solve the problem of excavating the shafts for basements. Each excavation takes 5 days. Management of the company decided to use four own excavators stored in four separated garages (they are moved to destinations each day).
  - The objective is to allocate each excavator to exactly one excavation with minimal cost. Since the costs are derived from distances (in km, see Table 8) between garages and destinations, we can concentrate only on these distances to define the objective.

Tab. 12 – Distances between garages and destinations

	Michle	Prosek	Radlice	Troja
Garage 1	5	22	12	18
Garage 2	15	17	6	10
Garage 3	8	25	5	20
Garage 4	10	12	19	12

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Assignment Problem

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if the excavator from garage } i \text{ goes to destination } j \\ 0 & \text{otherwise} \end{cases}$$

Linear Programming

Assignment Problem

- **Mathematical model**

Minimize $z = 5x_{11} + 22x_{12} + ... + 12x_{44}$

subject to

$x_{11} + x_{12} + x_{13} + x_{14} = 1$ (Garage 1)

$x_{21} + x_{22} + x_{23} + x_{24} = 1$ (Garage 2)

$x_{31} + x_{32} + x_{33} + x_{34} = 1$ (Garage 3)

$x_{41} + x_{42} + x_{43} + x_{44} = 1$ (Garage 4)

$x_{11} + x_{21} + x_{31} + x_{41} = 1$ (Michle)

$x_{12} + x_{22} + x_{32} + x_{42} = 1$ (Prosek)

$x_{13} + x_{23} + x_{33} + x_{43} = 1$ (Radlice)

$x_{14} + x_{24} + x_{34} + x_{44} = 1$ (Troja)

Linear Programming

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Assignment Problem

- **Optimal solution**
  - Decision variables

Tab. 13 – Optimal assignment of excavators to destinations

	Michle	Prosek	Radlice	Troja
Garage 1	1	0	0	0
Garage 2	0	0	0	1
Garage 3	0	0	1	0
Garage 4	0	1	0	0

  - Objective value

$$z_0 = 32$$

Linear Programming

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Assignment Problem

- Balanced problem

$$z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \rightarrow \min$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,2,\ldots,n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,2,\ldots,n$$

$$x_{ij} \in \{0,1\}, \quad i = 1,2,\ldots,n;\ j = 1,2,\ldots,n$$

- Unbalanced problem ($m > n$)

$$z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \rightarrow \min$$

$$\sum_{j=1}^{n} x_{ij} \leq 1, \quad i = 1,2,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} = 1, \quad j = 1,2,\ldots,n$$

$$x_{ij} \in \{0,1\}, \quad i = 1,2,\ldots,m;\ j = 1,2,\ldots,n$$

or using ($m - n$) dummy items

Linear Programming

Covering Problem

- In covering problem, we have a defined set of items (projects, jobs, processes, activities etc.) that have to be satisfied (covered) by some of several possible alternatives (firms, employees, managers etc.).
- Mostly, alternatives are selected based on the costs.

Linear Programming

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Covering Problem

- **Example**
  - In two of the six city districts it is necessary to establish emergency ambulance stations to cover all districts.
  - Table 14 shows the average arrival times (in min) from the station, established in a given district (at a predetermined location), to emergency incidents in each district..
  - The last row of the table shows the average daily frequency of emergency operations in each district.
  - The objective is to suggest where to establish stations and to assign the districts to be served by these stations so that the average daily operation time is minimal.

Tab. 14 – Statement of covering problem

	D_1	D_2	D_3	D_4	D_5	D_6
D_1	4	12	14	17	11	9
D_2	20	7	10	19	24	16
D_3	21	13	5	8	11	15
D_4	9	12	14	3	8	18
D_5	17	25	13	10	6	16
D_6	13	8	9	15	10	5
frequency	30	50	42	36	24	28

Linear Programming

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Covering Problem

- **Decision variables**

$$y_i = \begin{cases} 1 & \text{if in } i-\text{th district there is established} \\ & \text{the emergency station,} \\ 0 & \text{otherwise,} \end{cases} \qquad i = 1, 2, \ldots, n,$$

$$x_{ij} = \begin{cases} 1 & \text{if the station in } i-\text{th district} \\ & \text{serves } j-\text{th district,} \\ 0 & \text{otherwise,} \end{cases} \qquad \begin{array}{l} i = 1, 2, \ldots, n, \\ j = 1, 2, \ldots, n. \end{array}$$

Linear Programming

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Covering Problem

- **Objective**

$$z = \sum_{i=1}^{n} \sum_{j=1}^{n} f_j c_{ij} x_{ij} \rightarrow \min$$

- **Constraints**

$$\sum_{j=1}^{n} x_{ij} \leq (n-K+1) y_i , \quad i = 1, 2, \ldots, n$$

The station establishment and the districts served by this station.

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n$$

Each district will be served by exactly one station.

$$\sum_{i=1}^{n} y_i = K$$

Exactly K stations will be established.

Linear Programming

Covering Problem

- **Solution**

Tab. 15 – Optimal solution to covering problem

	D_1	D_2	D_3	D_4	D_5	D_6
D_1	0	0	0	0	0	0
D_2	0	0	0	0	0	0
D_3	0	0	0	0	0	0
D_4	1	0	0	1	1	0
D_5	0	0	0	0	0	0
D_6	0	1	1	0	0	1

The first station will be established in district D_4 and will serve (in addition to itself) districts D_1 and D_5.
The second one will be established in D_6 and will serve (in addition to itself) districts D_2 and D_3.

Linear Programming

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Traveling Salesman Problem

- **Definition of the problem**
  - Set of customers.
  - Each customer must be visited exactly once.
  - Cyclical route starts and ends in the home city (index 1).
  - Evaluation of direct travel form location i to location j is denoted by c_{ij} (distance, time or cost).
  - Objective is to minimize total length of the route, total travel time or total travel cost.

Linear Programming

Traveling Salesman Problem

- **Example**
  - A sales representative of the brewery located in Velvary must visit 7 pubs in 7 cities.
  - In Table 16, distances (in km) correspond to direct links (roads) between cities. A dash indicates there is no direct road between cities.
  - The objective is to visit all pubs minimizing total length of the route.

Tab. 16 – Distances between cities on roads

Obec	Velv	Kra	Lib	Sla	Zlo	Vra	Bri	Velt
Velvary	0	8	-	13	10	-	12	9
Kralupy	8	0	6	16	-	-	-	4
Libcice	-	6	0	-	-	-	-	-
Slany	13	16	-	0	7	-	-	-
Zlonice	10	-	-	7	0	7	13	-
Vrany	-	-	-	-	7	0	15	-
Briza	12	-	-	-	13	15	0	13
Veltrusy	9	4	-	-	-	-	13	0

Linear Programming

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Traveling Salesman Problem

- **Example**
  - Table 17 contains distances between all pairs of cities.

Tab. 17 – Distances between cities

Obec	Velv	Kra	Lib	Sla	Zlo	Vra	Bri	Velt
Velvary	0	8	14	13	10	17	12	9
Kralupy	8	0	6	16	18	25	17	4
Libcice	14	6	0	22	24	31	23	10
Slany	13	16	22	0	7	14	20	20
Zlonice	10	18	24	7	0	7	13	19
Vrany	17	25	31	14	7	0	15	26
Briza	12	17	23	20	13	15	0	13
Veltrusy	9	4	10	20	19	26	13	0

Linear Programming

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Traveling Salesman Problem

- **Feasible solution (Nearest Neighbor Algorithm)**
  1. Select any location as the initial one of the route (home city).
  2. Find the nearest location (not selected before) to the last location on the route and add it to the route. If it is impossible (all locations have been selected) then add the initial location to the route and go to Step 4.
  3. Go to step 2.
  4. End.

Tab. 18 – Feasible solution, Nearest Neighbor Algorithm

Obec	Velv	Kra	Lib	Sla	Zlo	Vra	Bri	Velt
Velvary	0	8	14	13	10	17	12	9
Kralupy	8	0	6	16	18	25	17	4
Libcice	14	6	0	22	24	31	23	10
Slany	13	16	22	0	7	14	20	20
Zlonice	10	18	24	7	0	7	13	19
Vrany	17	25	31	14	7	0	15	26
Briza	12	17	23	20	13	15	0	13
Veltrusy	9	4	10	20	19	26	13	0

- Objective value

$$z = 85$$

Linear Programming

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Traveling Salesman Problem

- **Decision variables**

$$x_{ij} = \begin{cases} 1 & \text{if a vehicle travels directly} \\ & \text{between nodes } i \text{ and } j \\ 0 & \text{otherwise} \end{cases} \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, n \end{array}$$

- **Dummy variables**

$$u_i = \text{dummy variable in sub-tours eliminating constraints}$$

Linear Programming

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Traveling Salesman Problem

- **Mathematical model**

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n$$

$$u_i + 1 - (n-1)(1 - x_{ij}) \leq u_j \quad \text{for } \begin{array}{l} i = 1, 2, \ldots, n \\ j = 2, 3, \ldots, n \end{array}$$

$$x_{ij} \in \{0, 1\} \quad \text{for } \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, n \end{array}$$

$$u_i \in \mathbb{R}_+ \quad \text{for } i = 1, 2, \ldots, n$$

Linear Programming

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Traveling Salesman Problem

- **Optimal solution**

Tab. 19 – Optimal route

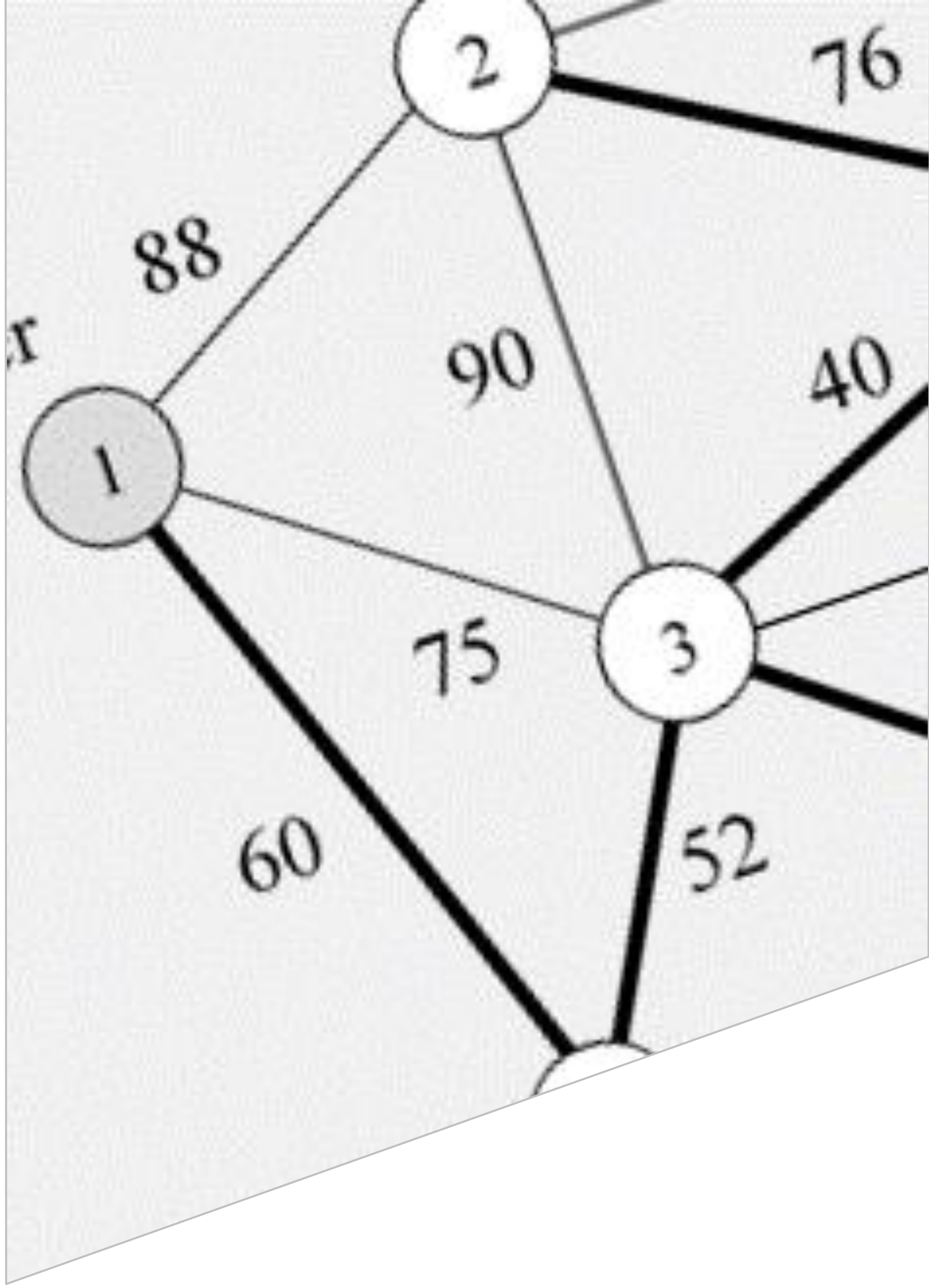
Obec	Velv	Kra	Lib	Sla	Zlo	Vra	Bri	Velt
Velvary	0	8	14	13	10	17	12	9
Kralupy	8	0	6	16	18	25	17	4
Libcice	14	6	0	22	24	31	23	10
Slany	13	16	22	0	7	14	20	20
Zlonice	10	18	24	7	0	7	13	19
Vrany	17	25	31	14	7	0	15	26
Briza	12	17	23	20	13	15	0	13
Veltrusy	9	4	10	20	19	26	13	0

- Objective value

$$z_0 = 79$$

3

Graph Modeling



Graph Modeling

Introduction

- **Seven bridges of Königsberg**

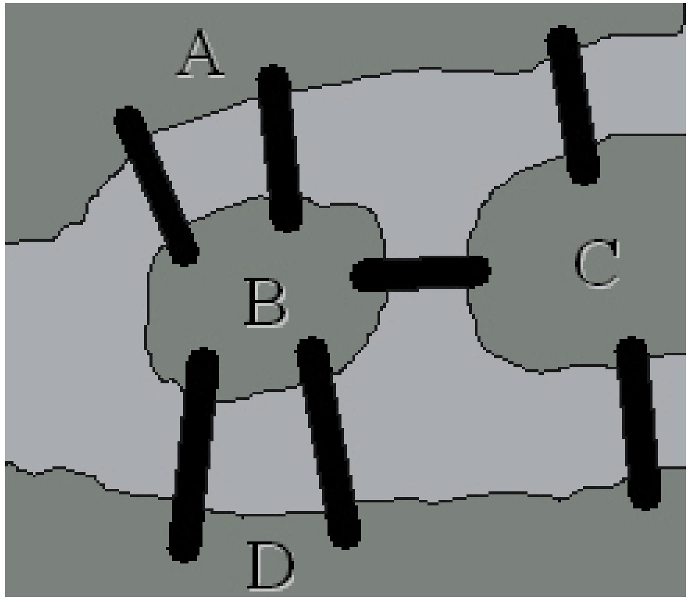


Fig. 7 – Real situation

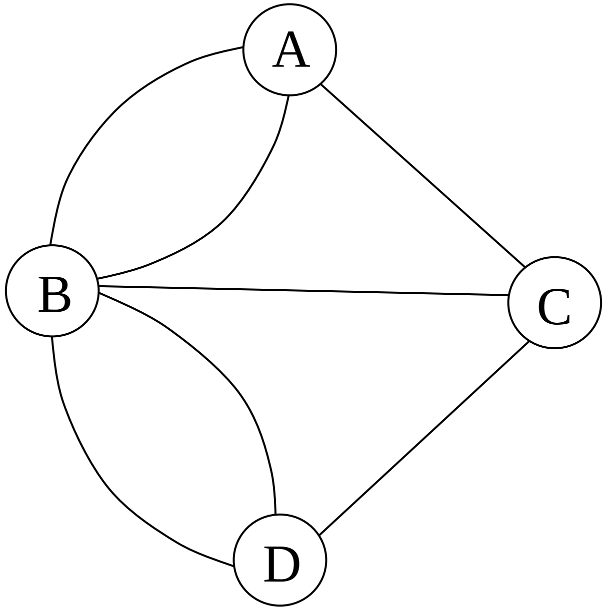


Fig. 8 – Graph as the model

Graph Modeling

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Introduction

- **Basic terminology**

  Graph is a set $G = \{V, E\}$, where V is a set of vertices (nodes) and E is a set of edges (arcs).
  Undirected arc is a set of two vertices $\{i, j\}$.
  Directed arc is an ordered pair of two vertices (i, j).
  In undirected graph all arcs are undirected.
  In directed graph (digraph) all arcs are directed.
  Mixed graph contains both undirected and directed arcs.
  Two nodes that are contained in an arc are adjacent.
  Two arcs that share a node are adjacent.
  An arc and a node contained in that arc are incident.
  Degree of a node (in undirected graph) is a number of incident arcs.
  In-degree of a node (in directed graph) is a number of incident arcs in which the node is the terminal one.
  Out-degree of a node (in directed graph) is a number of incident arcs in which the node is the initial one.

Graph Modeling

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Introduction

- **Basic terminology**

Walk from node i to node j is a sequence of nodes and arcs, where i is the initial node and j is the terminal node (nodes and arcs may be repeated).
Trail is a walk with no repeated arc.
Path is a trail with no repeated node.
Cycle is closed walk (the initial node is the terminal one).
In directed path (in directed graph) a direction of all arcs is respected.
In undirected path (in directed graph) a direction of all arcs may not be respected.
Undirected graph is connected if between each pair of nodes there is a path.
Directed graph is connected if there is a directed or undirected path between each pair of nodes.

Graph Modeling

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Introduction

- **Basic terminology**

Directed graph is strongly connected if there is a directed path between each pair of nodes.
Undirected graph is complete if there is an arc between each pair of nodes.
Tree is a connected undirected graph with no cycles.
Subgraph of graph $G = \{V, E\}$ is a graph $G' = \{V', E'\}$, where $V' \subseteq V$ and $E' \subseteq E$.
Spanning tree of the graph G is a subgraph G', where $V' = V$ and which is a tree.
Valued graph has numbers associated with nodes or/and arcs.
Hamiltonian cycle is a cycle that includes each node of the graph exactly once.
Eulerian cycle includes each arc of the graph exactly once.
Eulerian trail is a trail that includes each arc of the graph.
Eulerian graph is a graph in which the Eulerian cycle can be found.

Graph Modeling

Introduction

- **Basic terminology**
  - Connected and unconnected graph

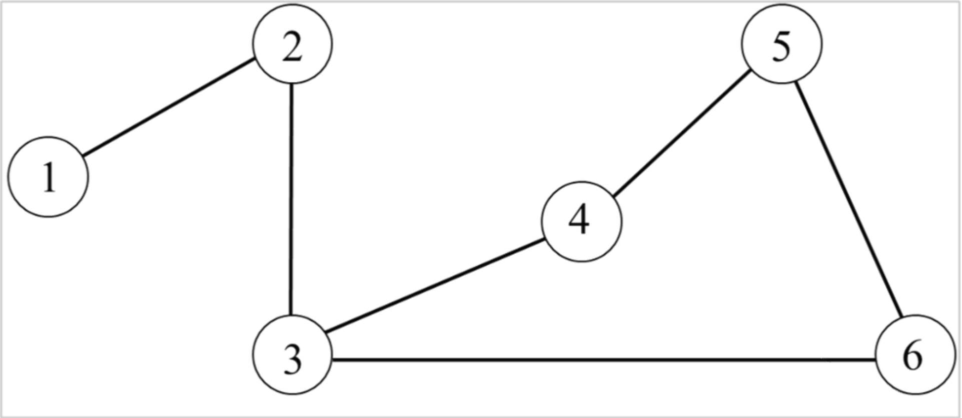

Fig. 9 – Connected graph

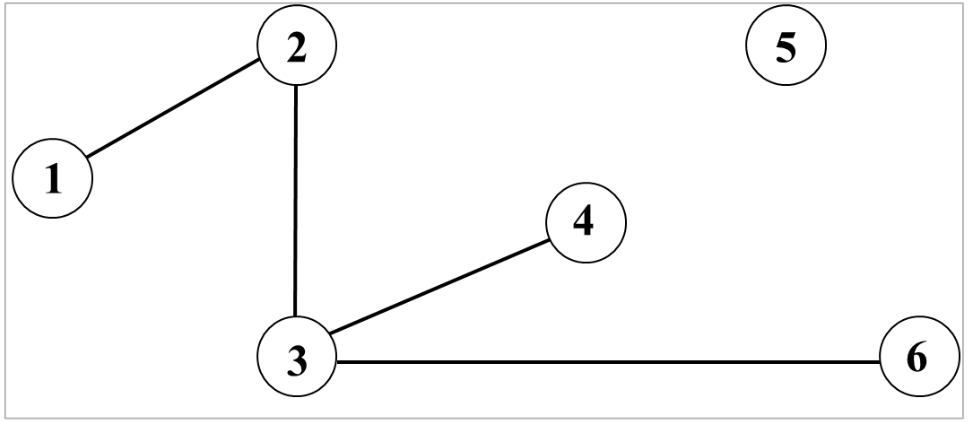

Fig. 10 – Unconnected graph

INTERNAL

Graph Modeling

Introduction

- **Basic terminology**
  - Cycle (circuit)

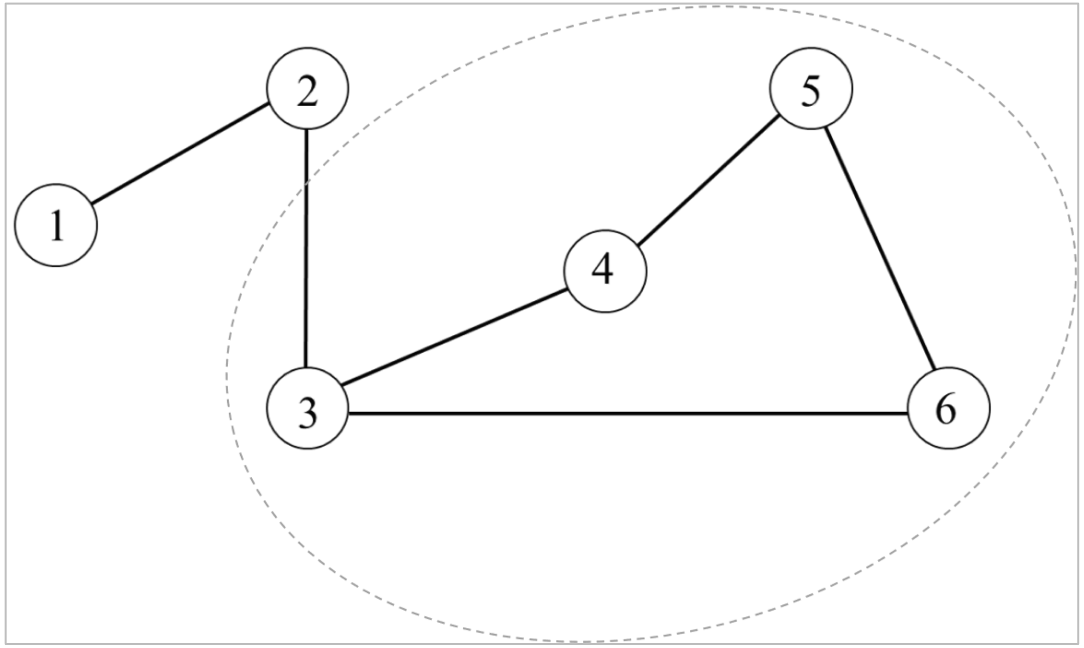

Fig. 11 – Cycle

Graph Modeling

Introduction

- **Basic terminology**
  - Tree and spanning tree

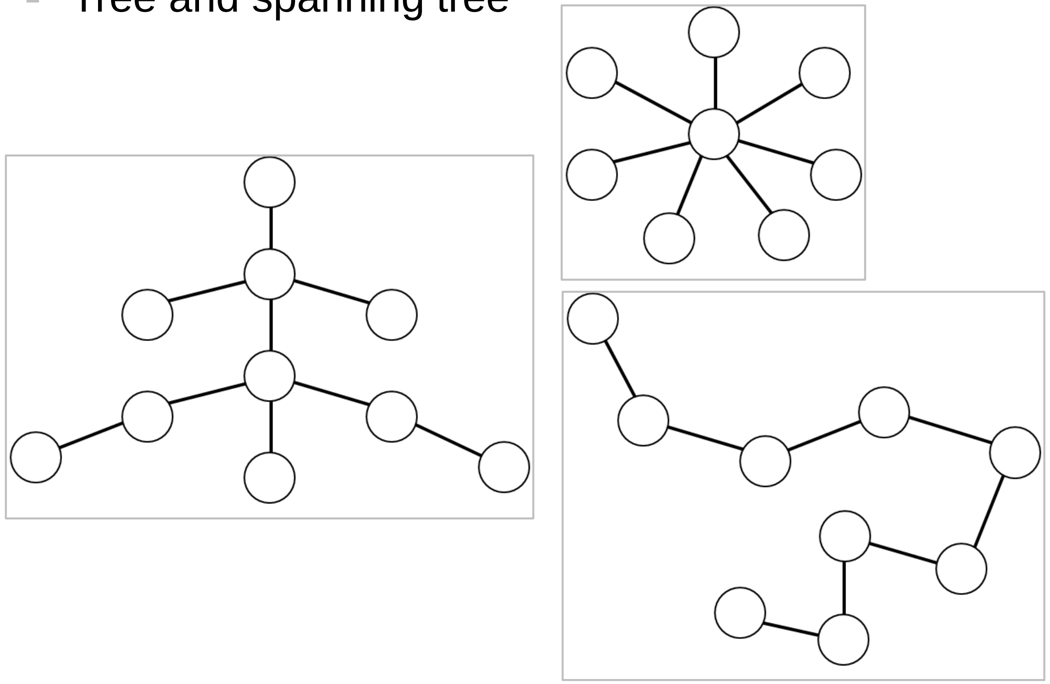

Fig. 12 – Trees

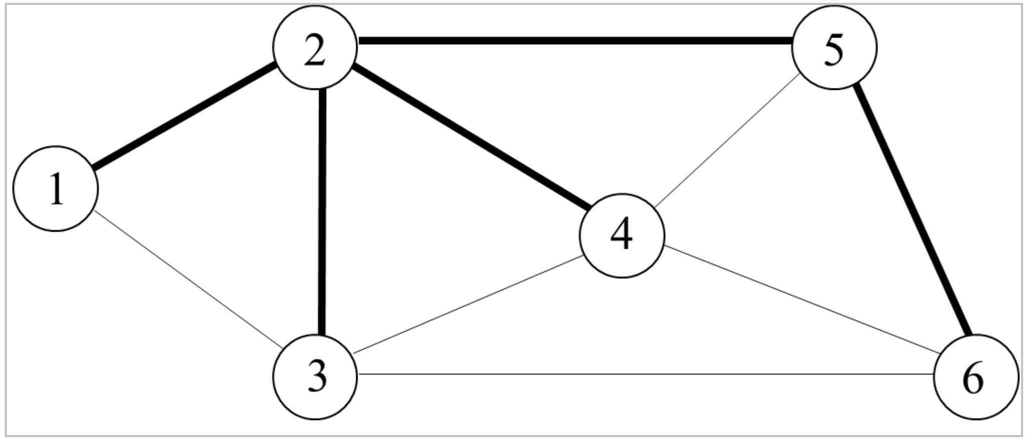

Fig. 13 – Spanning tree

Graph Modeling

Shortest Path Problem

- The objective is to find the shortest path between a pair of nodes.
- The problem while using of GPS navigation in the car or with search of connection on a map portal.
- Many algorithms are designed specially to find distances between all pairs of nodes in given graph.
- Dijkstra’s algorithm is designed to find the shortest path from one particular node to all other nodes in the graph.

Graph Modeling

Shortest Path Problem

- **Example**

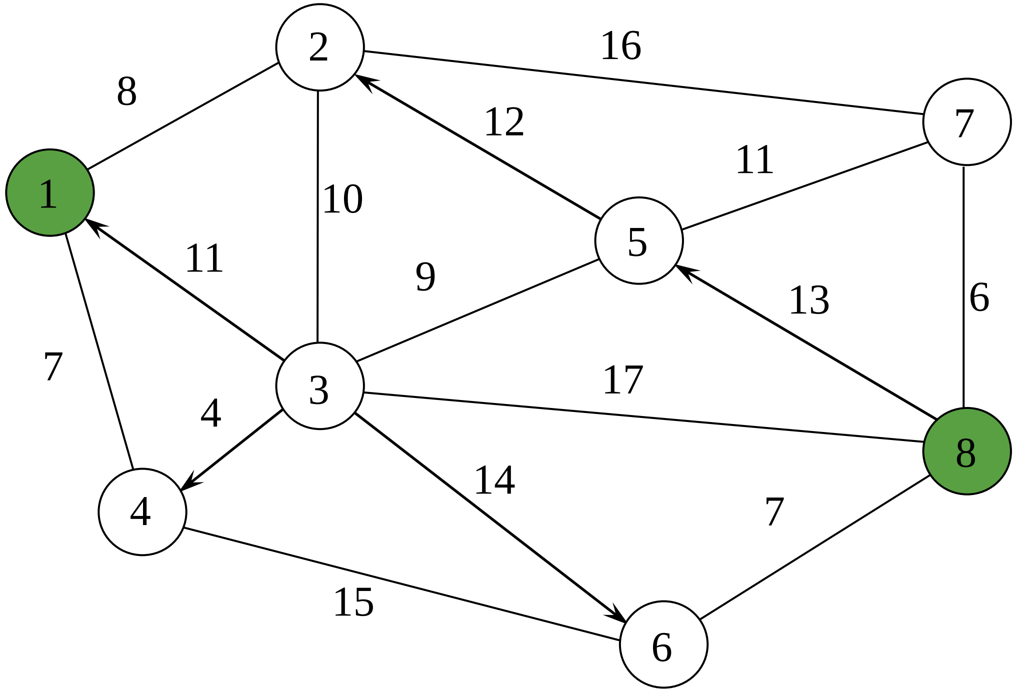

Fig. 14 – Distribution network for the transport of overweight load

Graph Modeling

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Shortest Path Problem

- In the first step, it is necessary to transcript the graph into the Table 20, where are node numbers (first row), calculated distances – reserved for later (second row) and the table list (for each node i, all nodes j to which an arc leads from it).

Tab. 20 – Input data for calculation

i	1	2	3	4	5	6	7	8
t_i	0							
$j(y_{ij})$	2(8)	3(10)	2(10)	6(15)	2(12)	4(15)	2(16)	3(17)
	4(7)	7(16)	4(4)		3(9)	8(7)	5(11)	5(13)
			5(9)		7(11)		8(6)	6(7)
			6(14)					7(6)
			8(17)					

- In the following steps, we will consider only the columns in which value t_i is already calculated. We always look for the shortest paths from just these nodes to all nodes j whose value t_i is not yet determined, through the arcs listed.

Graph Modeling

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Shortest Path Problem

- In the second step, we will consider only the first column and nodes 2 and 4.
- There are two direct paths from node 1: to node 2, which length is (0+8), and to node 4, which length is (0+7). We take the minimum of these two values i.e., 7, and write this value to the corresponding node i.e., 4. Thus, $t_4 = 7$. The selected value (arc) is framed and scratched are those arcs ending in currently selected node.

Tab. 21 – Calculation of the shortest path to node 4

i	1	2	3	4	5	6	7	8
t_i	0			7				
$j(y_{ij})$	2(8)	3(10)	2(10)	6(15)	2(12)	4(15)	2(16)	3(17)
	4(7)	7(16)	4(4)		3(9)	8(7)	5(11)	5(13)
			5(9)		7(11)		8(6)	6(7)
			6(14)					7(6)
			8(17)					

INTERNAL

Graph Modeling

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Shortest Path Problem

- In the third step, based on Table 21, we look for the minimum of two values: (0+8) and (7+15). In the next step, we calculate min(8+10, 8+16, 7+15) = 8+10 = 18.

Tab. 22 – Calculation of the shortest path to node 2

i	1	2	3	4	5	6	7	8
t_i	0	8		7				
$j(y_{ij})$	2(8)	3(10)	2(10)	6(15)	2(12)	4(15)	2(16)	3(17)
	4(7)	7(16)	4(4)		3(9)	8(7)	5(11)	5(13)
			5(9)		7(11)		8(6)	6(7)
			6(14)					7(6)
			8(17)					

- We continue in this process till all values t_i are determined.

Graph Modeling

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Shortest Path Problem

Tab. 23 – Final solution for the shortest path

i	1	2	3	4	5	6	7	8
t_i	0	8	18	7	27	22	24	29
$j(y_{ij})$	2(8)	3(10)	~~2(10)~~	6(15)	~~2(12)~~	~~4(15)~~	~~2(16)~~	~~3(17)~~
	4(7)	7(16)	~~4(4)~~		~~3(9)~~	8(7)	~~5(11)~~	5(13)
			5(9)		~~7(11)~~		~~8(6)~~	~~6(7)~~
			~~6(14)~~					~~7(6)~~
			~~8(17)~~					

Graph Modeling

Shortest Path Problem

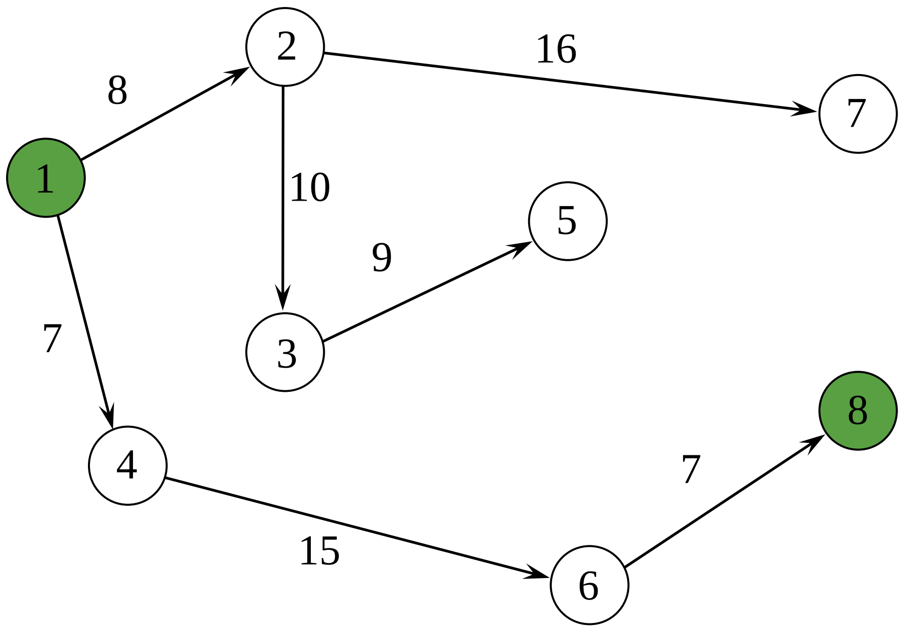

Fig. 15 – Shortest paths from node 1 to all other nodes

Graph Modeling

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Minimal Spanning Tree

- **Definition of the problem**
  - Graph with n nodes and valued arcs.
  - Objective is to find the spanning tree with the minimal sum of values assigned to selected arcs.
- **Optimization method**
  1. Sort arcs in ascending (non-descending) order according to values.
  2. Select two arcs with minimal values.
  3. Select the arc (in the ordered list) with the minimal value.
  4. If the arc creates a cycle with any of selected arcs, skip the arc, otherwise add the arc to the set of selected arcs.
  5. If the number of selected arcs is equal to $n - 1$ then go to step 6, otherwise go to step 3.
  6. End.

Graph Modeling

Minimal Spanning Tree

- **Example**
  - The managerial problem is to connect 9 locations of an exhibition area with the source of electricity power.
  - The objective is to minimize the cost of all the extensions. The direct distances (in meters) between locations can be found in figure 16.
  - The node 1 is the source of power.
  - The price per meter of a cable is 10 CZK.

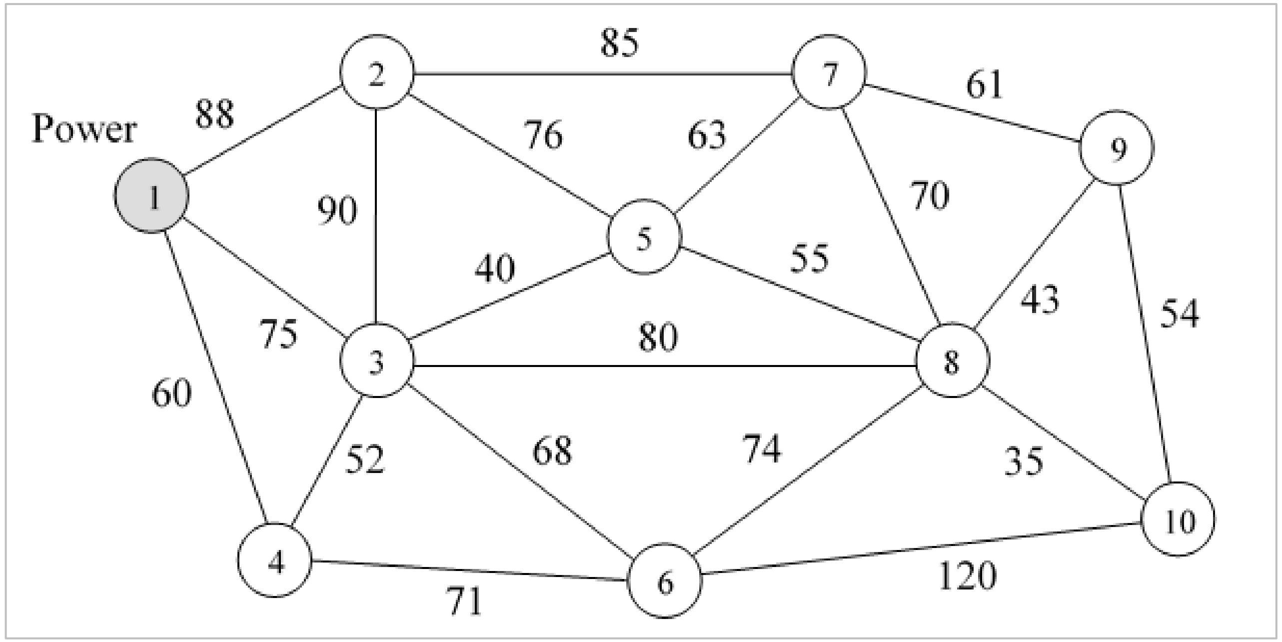

Fig. 16 – Map of the exhibition area

Graph Modeling

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Minimal Spanning Tree

- **Optimal solution**

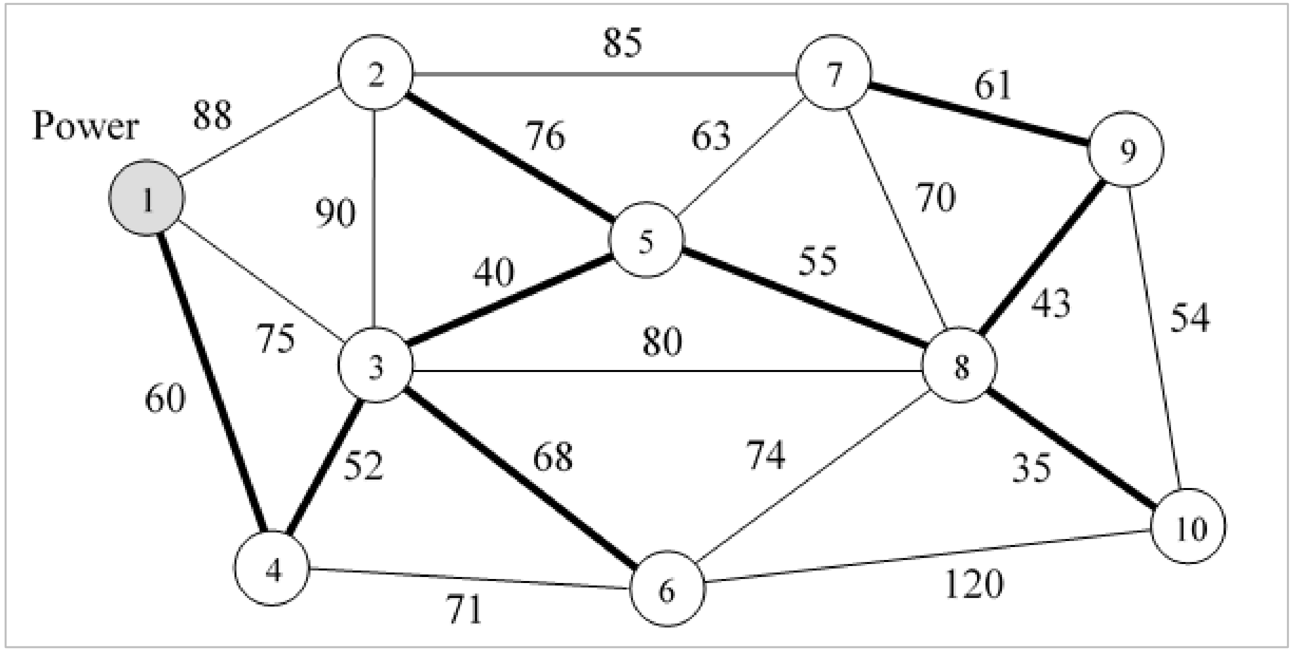


Fig. 17 – Cable placement

- Minimal value of spanning tree

  $z_0 = 490$

INTERNAL

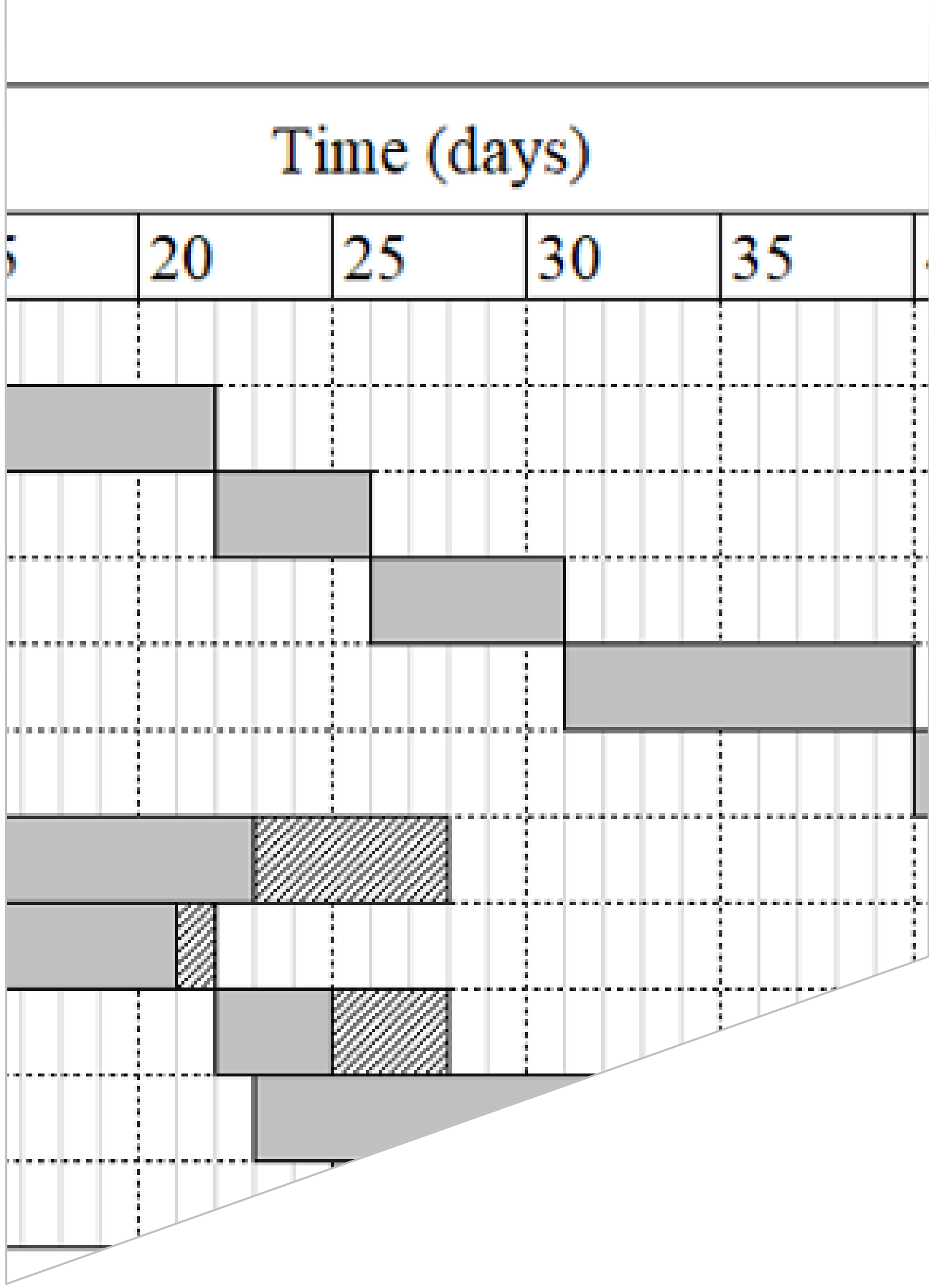
Graph Modeling

Other optimization problems

- **Connection**
  - Minimal Spanning Tree.
  - Minimal Steiner Tree.
- **Paths and routes**
  - Minimal Path Problem.
  - Traveling Salesman Problem.
  - Vehicle Routing Problem.
  - Pick-up and Delivery Problem.
  - Chinese Postman Problem.
- **Flows**
  - Maximal Flow Problem.
  - Minimal Cost Flow Problem.
  - Transshipment Problem.

4

Project Management



Project Management

Introduction

- **Project**
  - Set of interrelated activities (jobs, operations).
  - Activity:
    - duration time,
    - cost,
    - resources,
    - immediate predecessors.
- **Duration time**
  - deterministic (constant) – Critical Path Method (CPM), Metra Potential Method (MPM)
  - probabilistic (random variable) – Program Evaluation and Review Technique (PERT)

Project Management

Introduction

- **Network**
  - Graphical representation of a project.
  - Activities are represented by
    - arcs (CPM),
    - nodes (MPM).

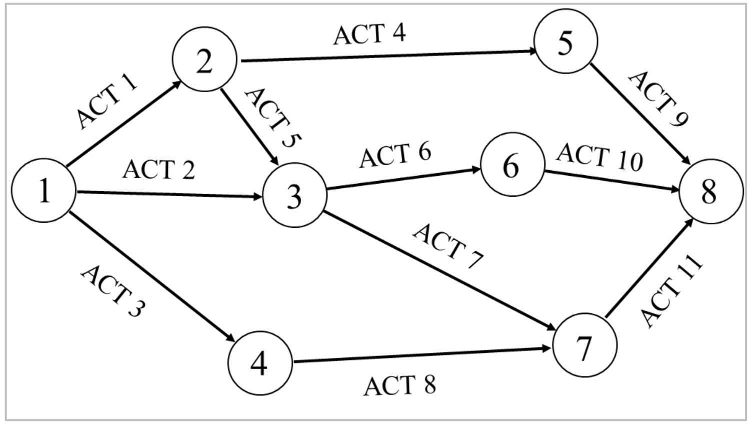


Fig. 18 – Project network represented by arcs

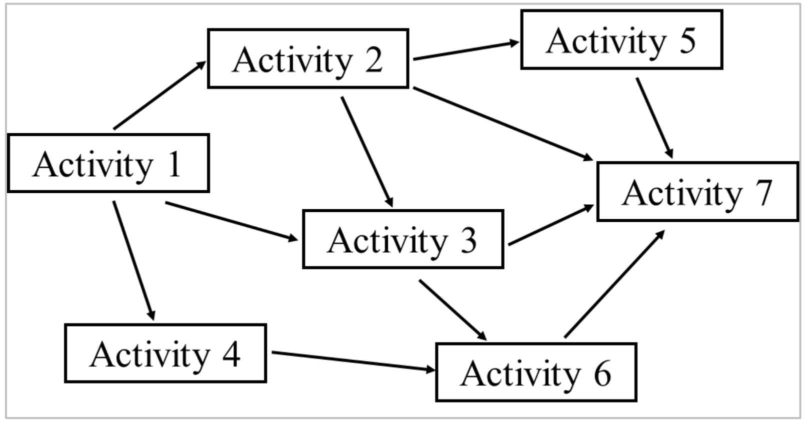


Fig. 19 – Project network represented by nodes

Project Management

Critical Path Method

- **Phases of project**
  - Planning
    - determination of all activities, their time durations and immediate predecessors,
    - construction of a network representing the project.
  - Scheduling
    - finish time of the project,
    - start and finish times of each activity,
    - a critical path consisting of critical activities,
    - non-critical activities and their possible delay,
    - Gantt chart.
  - Controlling
    - comparison of the real performance of the project with the proposed schedule,
    - dynamic changes in the schedule.

Project Management

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Critical Path Method

- **Example**
  - Project manager of direct marketing company has received the direction to prepare the Christmas compilation of Czech carols.
  - The offer should be sent to the addresses of company's customers, who are interested in carols. For this purpose, a thorough analysis should be carried out to select the best customers for the promotion.
  - The project manager must consider 11 activities to realize the project (see Table 13).
  - All the envelopes have to be delivered until December 12 (Thursday). Described activities together with their duration (in working days) and their immediate predecessors can be found in Table 13.
  - The project manager must now set the starting date of the project such that it will be finished exactly on the specified date.

Project Management

Critical Path Method

- **Example**

Tab. 24 – Activities of direct marketing project

Activity	Activity Description	Duration	Immediate Predecessors
A	Songs Selection	15	None
B	Mastering	8	A
C	Promotion Material Elaborating	6	A
D	Customers Analysis	7	A
E	Promotion Material Production	4	C, D
F	Promotion Material to Printing House	5	E
G	Customers Selection	3	D
H	Make CD Copies	12	B, D
I	Data to Printing House	3	G
J	Laser Print	9	F, I
K	Mailing	8	H, J

Project Management

Critical Path Method

- **Construction of the network**
  - One start node and one finish node.
  - Each activity must be represented just by one arc.
  - Two nodes are connected maximally by one arc.
  - The node, representing the completion of an activity, has higher number than the node representing the start of this activity – the rule prevents circuits in the network.
  - Dummy activity (arc) assures the correct interrelations between real activities. Its duration is zero

Project Management

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Critical Path Method

- **Construction of the network**

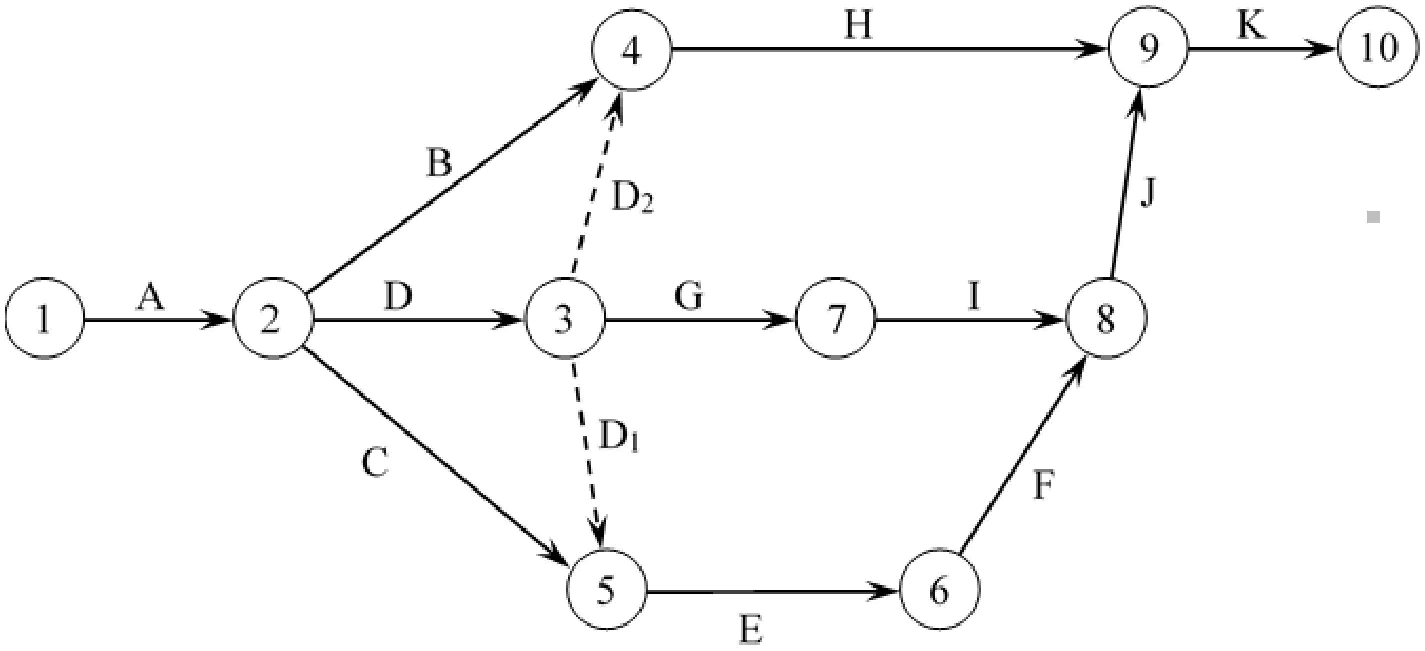


Fig. 20 – Network of direct marketing project

- D_1, D_2 – dummy activities

Project Management

Critical Path Method

- **Event analysis – forward pass**
  - The earliest event time for a node is the earliest time at which all the preceding activities have been completed.
  - The computation runs through the nodes according to their numbers (in ascending order), from the start of the project to its finish.

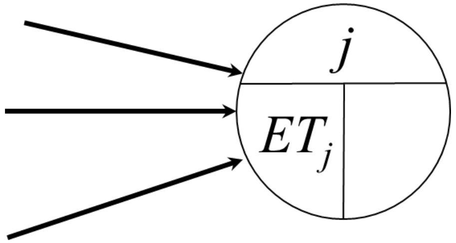


Fig. 21 – Earliest event time for a node

$$ET_j = \max_i (ET_i + t_{ij}) \qquad j = 2, 3, \ldots, n$$

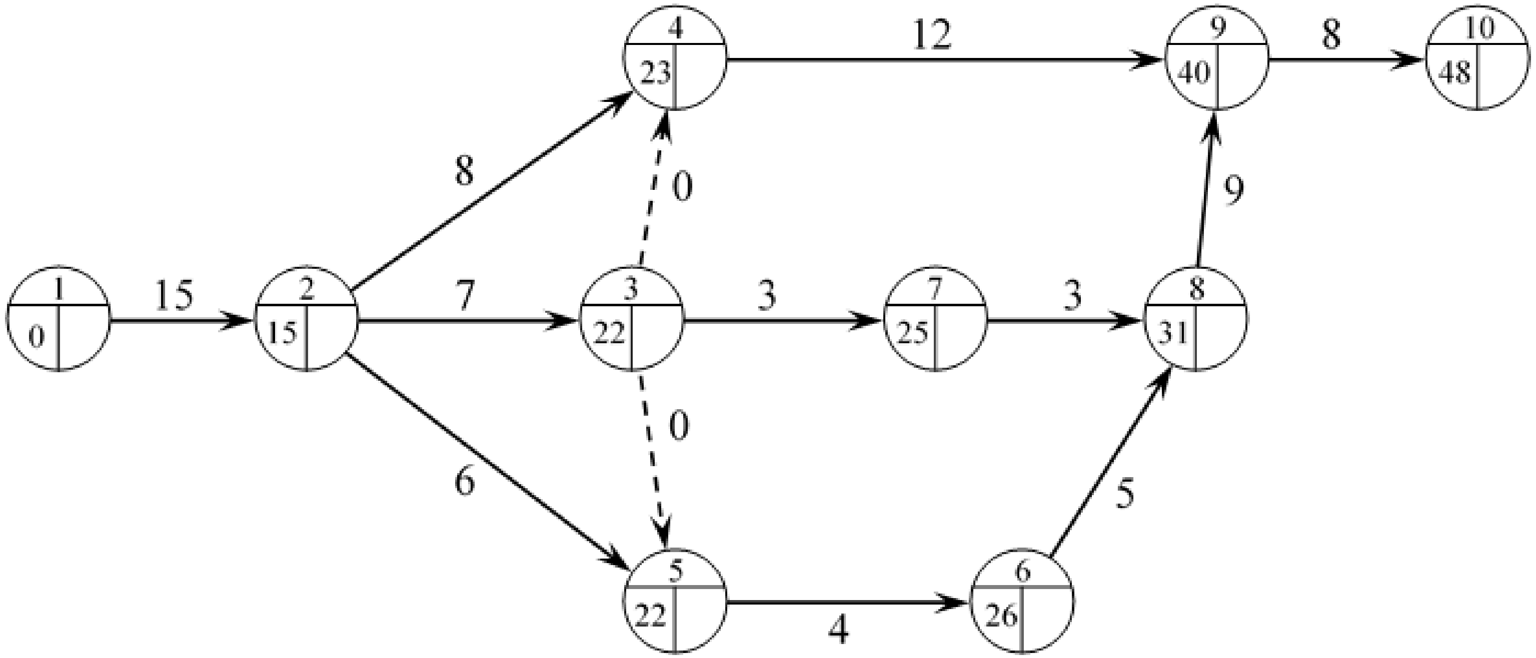
- The earliest event time for completing a project is the earliest time of the last event:

$$T = ET_n$$

Project Management

Critical Path Method

- **Event analysis – forward pass**



$T = 48$

Fig. 22 – Forward pass

Project Management

Critical Path Method

- **Event analysis – backward pass**
  - The latest event time for a node is the latest time at which the event can occur without delaying the determined completion time of the project.
  - Whereas the forward pass crosses the nodes in ascending order according to their numbers, backward pass goes from the finish of the project to its start in the opposite (descending) order.

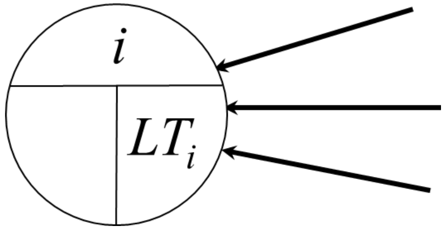


Fig. 23 – Latest event time for a node

- The latest event time for completing a project can be set in the planning process or can be considered as the earliest event time for completing a project:

$$LT_n = T_{PL} \geq T$$

$$LT_i = \min_j (LT_j - t_{ij}) \qquad i = n - 1, n - 2, \ldots, 1$$

Project Management

Critical Path Method

- **Event analysis – backward pass**

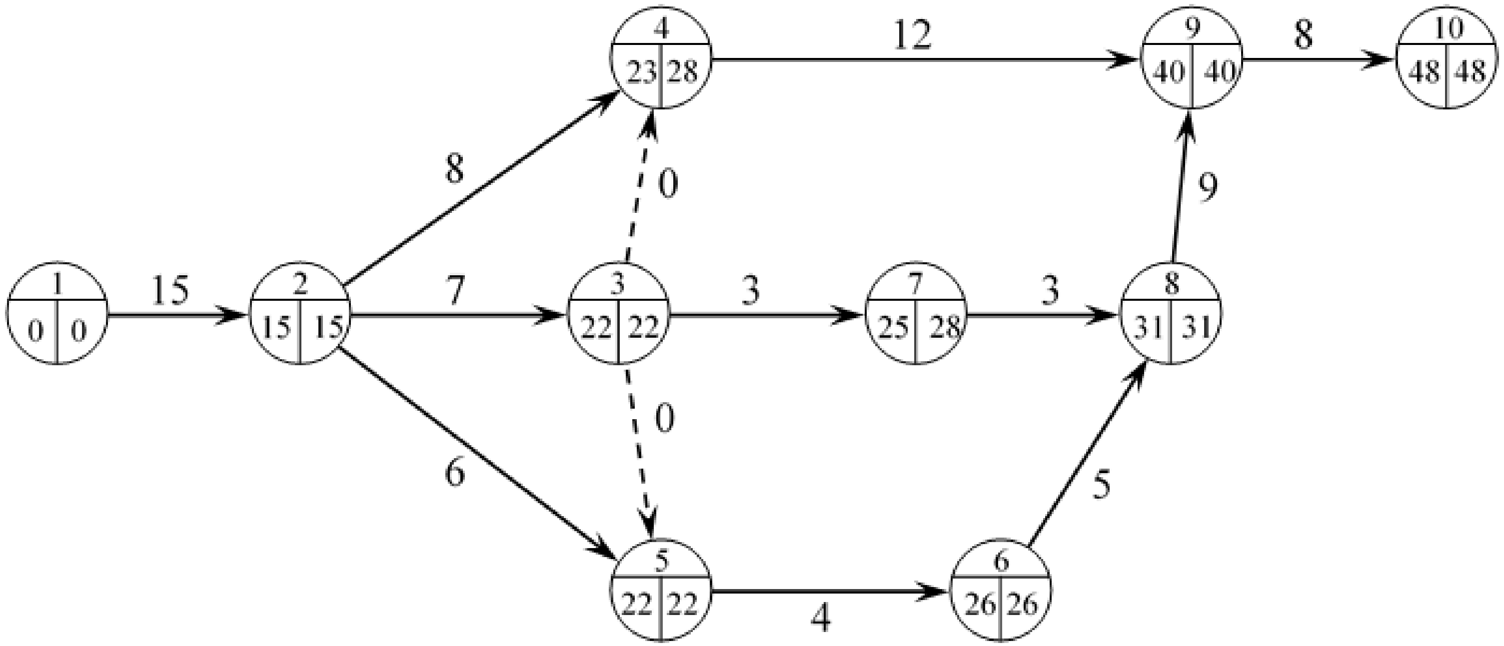


Fig. 24 – Backward pass

- Deadline T_{PL} = 48

Project Management

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Critical Path Method

- **Activity analysis**
  - Calculation of total floats (slacks) for all activities: $TF_{ij} = (LT_j - ET_i - t_{ij})$
  - Critical activities: $TF_{ij} = T_{PL} - T$
  - Non-critical activities: $TF_{ij} > T_{PL} - T$
  - Total floats give three possibilities of delay of each activity (to meet deadline T_{PL}) :
    1. A start of the activity can be postponed.
    2. Duration of the activity can be extended (the activity can be even interrupted).
    3. The combination of possibilities 1 and 2.

Project Management

Critical Path Method

- **Activity analysis**

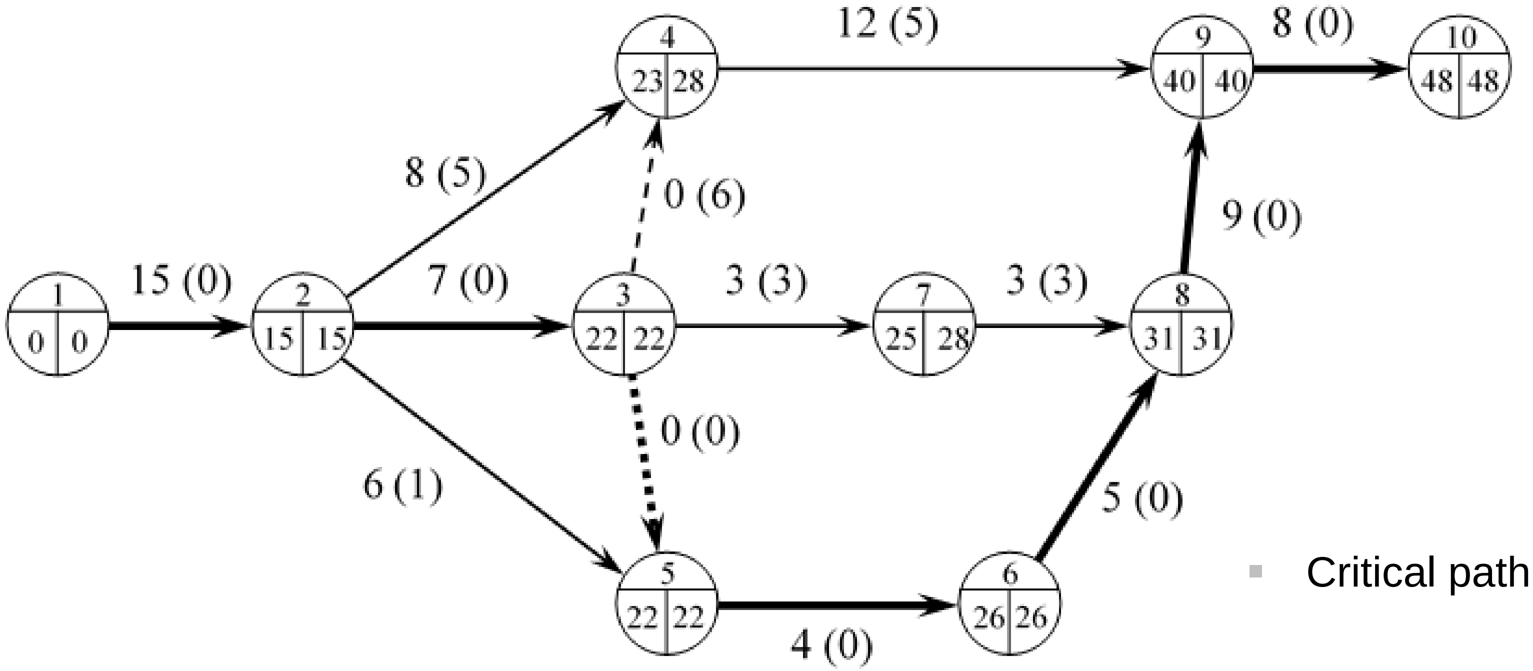

Fig. 25 – Critical path

- Critical path = $A – D – (D_1) – E – F – J – K$

Project Management

Critical Path Method

- **Gantt chart**

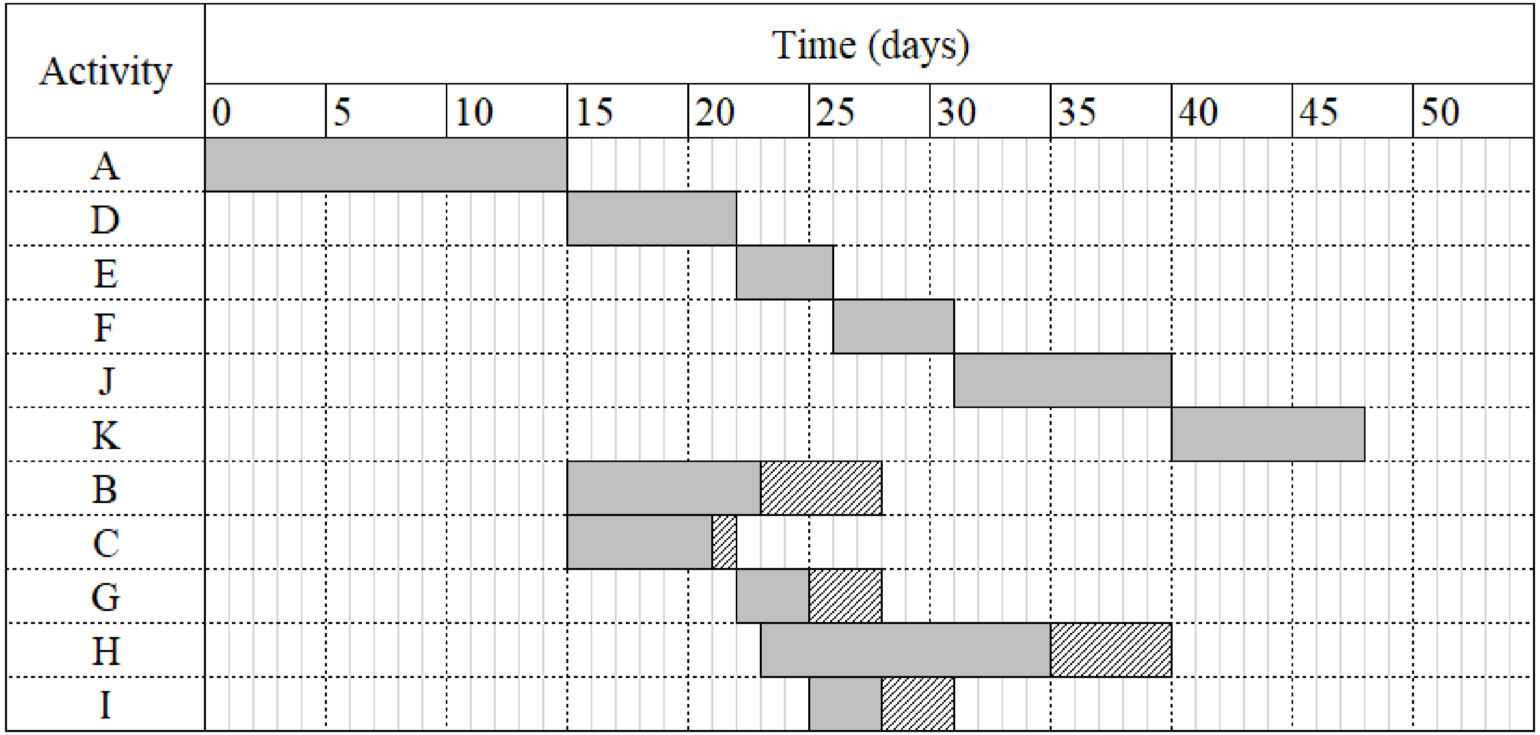

Fig. 26 – Gantt chart

- Deadline: December 12
- Start of the project: October 3

Project Management

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PERT (Program Evaluation and Review Technique)

- **Assumptions**
  - Probabilistic duration time of each activity.
  - β-distribution values:
    - a_{ij} - optimistic estimate as the shortest possible duration of the activity,
    - b_{ij} - pessimistic estimate as the longest duration,
    - m_{ij} - most likely estimate as the duration, which assumes most frequent conditions.
- **Mean of completion time**

$$\mu_{ij} = \frac{a_{ij} + 4m_{ij} + b_{ij}}{6}$$

- **Standard deviation of completion time**

$$\sigma_{ij} = \frac{b_{ij} - a_{ij}}{6}$$

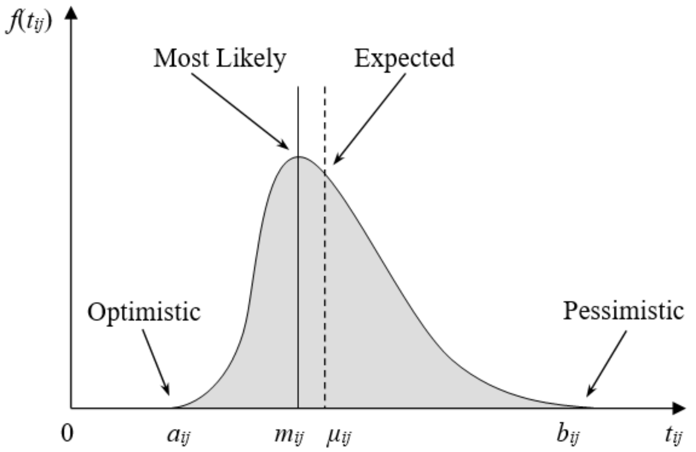

Fig. 27 – Density function of β-distribution

Project Management

PERT (Program Evaluation and Review Technique)

- **Example**
  - In project there are introduced the following estimations of activities duration times.

Tab. 25 – Direct marketing project – input data for the PERT

Activity	Activity Description	a_{ij}	m_{ij}	b_{ij}	Immediate Predecessors
A	Songs Selection	11	15	19	None
B	Mastering	7	8	9	A
C	Promotion Material Elaborating	5	6	7	A
D	Customers Analysis	5	7	9	A
E	Promotion Material Production	2	3	10	C, D
F	Promotion Material to Printing House	3	4	11	E
G	Customers Selection	2	3	4	D
H	Make CD Copies	8	11	20	B, D
I	Data to Printing House	2	3	4	G
J	Laser Print	6	8	16	F, I
K	Mailing	6	8	10	H, J

Project Management

PERT (Program Evaluation and Review Technique)

- **Method description**
  - Construction of the network.
  - Calculation of expected completion times (and standard deviations of completion times).
  - Application of the CPM, determination of critical path (CP).
  - **Expected project duration time**

$$M = \sum_{(i,j)\in \mathrm{CP}} \mu_{ij}$$

  - **Variance of the project duration time**

$$\sigma^2 = \sum_{(i,j)\in \mathrm{CP}} \sigma_{ij}^2$$

  - **Standard deviation of the project duration time**

$$\sigma = \sqrt{\sigma^2}$$

Project Management

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PERT (Program Evaluation and Review Technique)

- **Method description**
  - Probability analysis:
    - **What is the probability of project completion within a desired time T_D?**
      - Transformation to standard normal distribution $N(0,1)$:

$$z = \frac{T_D - M}{\sigma}$$

    - **What is the completion time T_D in which the project will be finished with a desired probability p?**

$$T_D = M + z_p\sigma$$

Project Management

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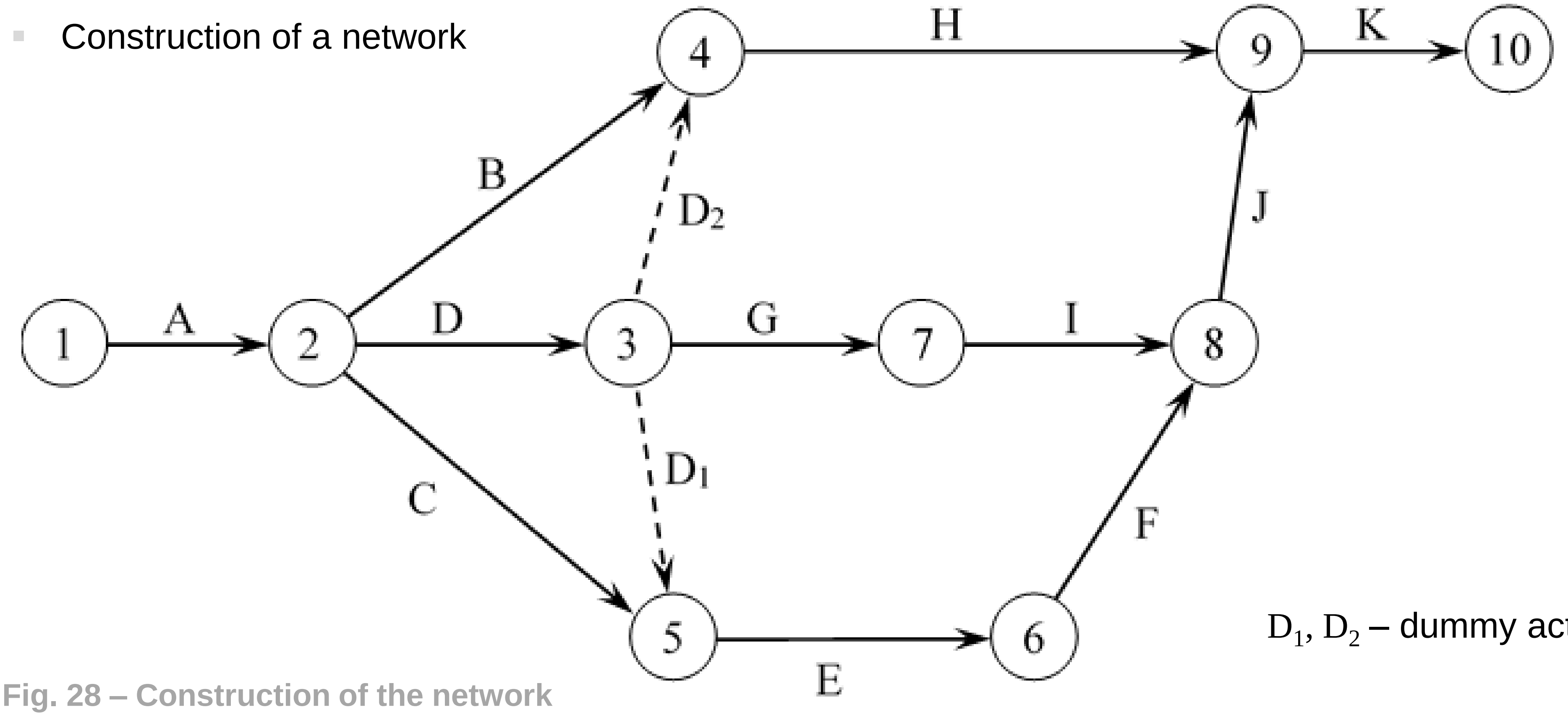
PERT (Program Evaluation and Review Technique)

- **Example**
  - Construction of a network



D_1, D_2 – dummy activities

Fig. 28 – Construction of the network

INTERNAL

Project Management

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PERT (Program Evaluation and Review Technique)

- **Example**
  - Calculation of the mean values of duration time

Tab. 26 – Direct marketing project – mean values of duration times

Activity	Activity Description	a_{ij}	m_{ij}	b_{ij}	μ_{ij}
A	Songs Selection	11	15	19	15
B	Mastering	7	8	9	8
C	Promotion Material Elaborating	5	6	7	6
D	Customers Analysis	5	7	9	7
E	Promotion Material Production	2	3	10	4
F	Promotion Material to Printing House	3	4	11	5
G	Customers Selection	2	3	4	3
H	Make CD Copies	8	11	20	12
I	Data to Printing House	2	3	4	3
J	Laser Print	6	8	16	9
K	Mailing	6	8	10	8

Project Management

PERT (Program Evaluation and Review Technique)

- **Example**
  - Application of the CPM

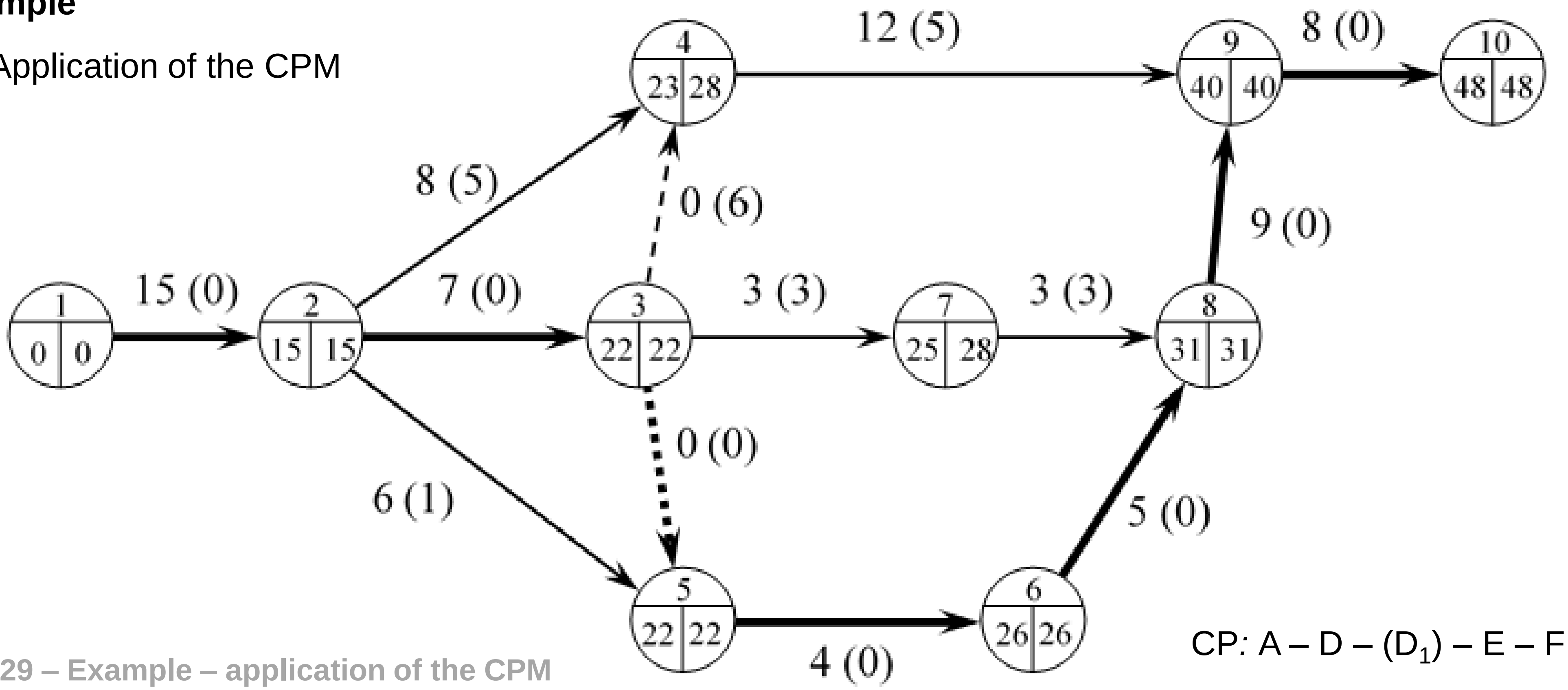

Fig. 29 – Example – application of the CPM

CP: A – D – (D_1) – E – F – J – K

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Project Management

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PERT (Program Evaluation and Review Technique)

- **Example**
  - Calculation of the standard deviations of duration time

Tab. 27 – Example

Activity	Activity Description	a_{ij}	m_{ij}	b_{ij}	μ_{ij}	σ_{ij}
A	Songs Selection	11	15	19	15	8/6
D	Customers Analysis	5	7	9	7	4/6
E	Promotion Material Production	2	3	10	4	8/6
F	Promotion Material to Printing House	3	4	11	5	8/6
J	Laser Print	6	8	16	9	10/6
K	Mailing	6	8	10	8	4/6

Project Management

PERT (Program Evaluation and Review Technique)

- **Example**
  - Expected project duration time

    $$M = 15 + 7 + 4 + 5 + 9 + 8 = 48$$

  - Variance of the project duration time

    $$\sigma^2 = (8/6)^2 + (4/6)^2 + (8/6)^2 + (8/6)^2 + (10/6)^2 + (4/6)^2 = 9$$

  - Standard deviation of the project duration time

    $$\sigma = 3$$

Project Management

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PERT (Program Evaluation and Review Technique)

- **Example – probability analysis**
  - What is the probability of project completion within a desired time T_D = 45 days?

$$z = \frac{45 - 48}{3} = -1 \xrightarrow{\text{Table N(0,1)}} p = 0.1587$$

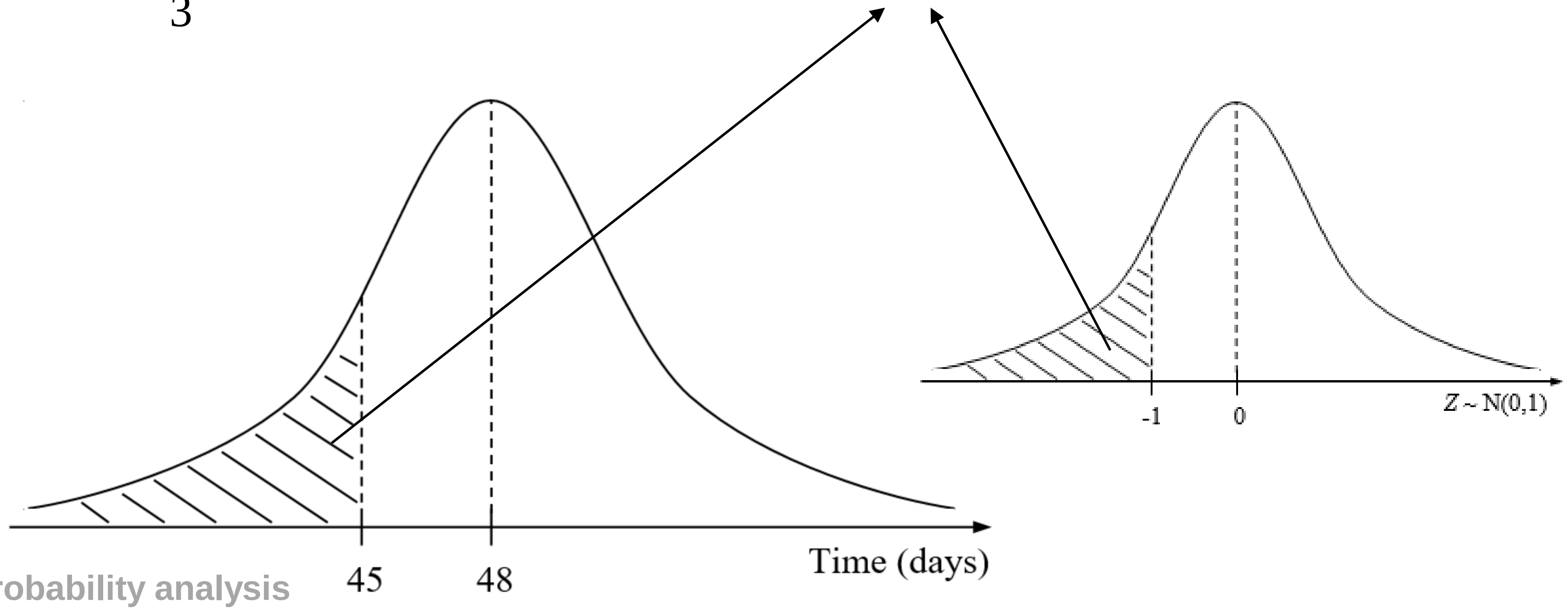

Fig. 30 – Probability analysis

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Project Management

PERT (Program Evaluation and Review Technique)

- **Example – probability analysis**
  - Set a deadline by which the project will have been completed with 95% confidence.

$$p = 0.95 \xrightarrow{\text{Table N(0,1)}} z_{0.95} = 1.64 \longrightarrow T_D = 48 + 1.64(3) = 52.92$$

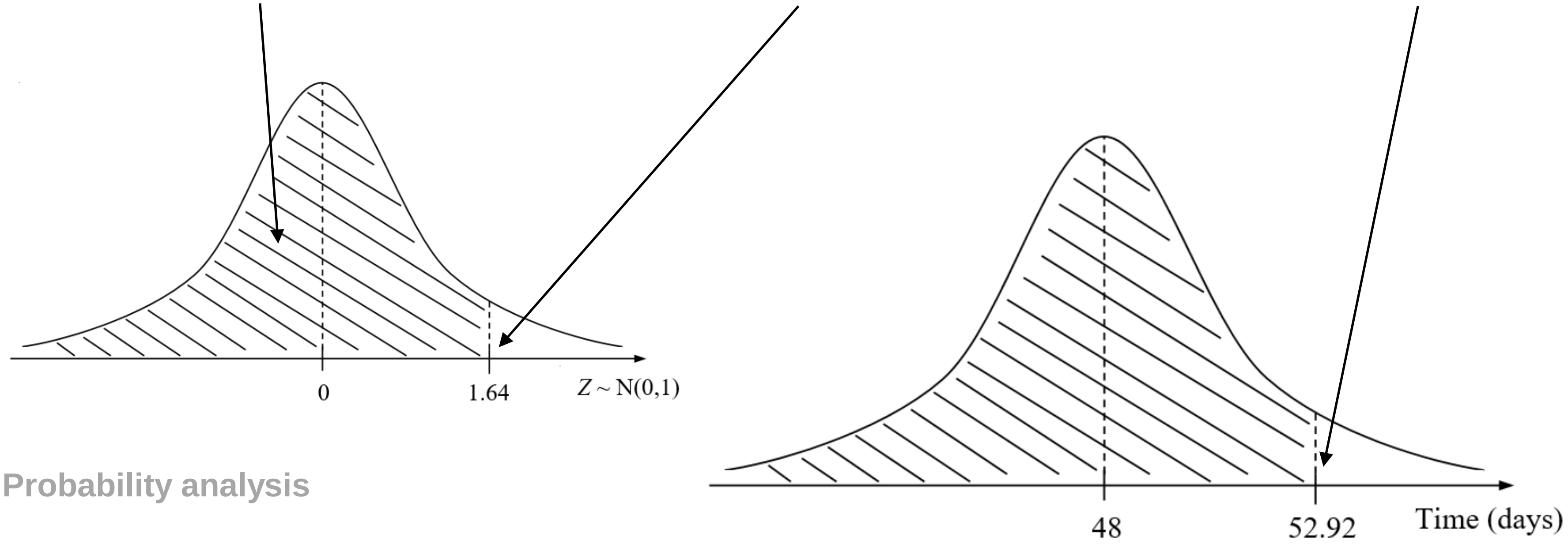
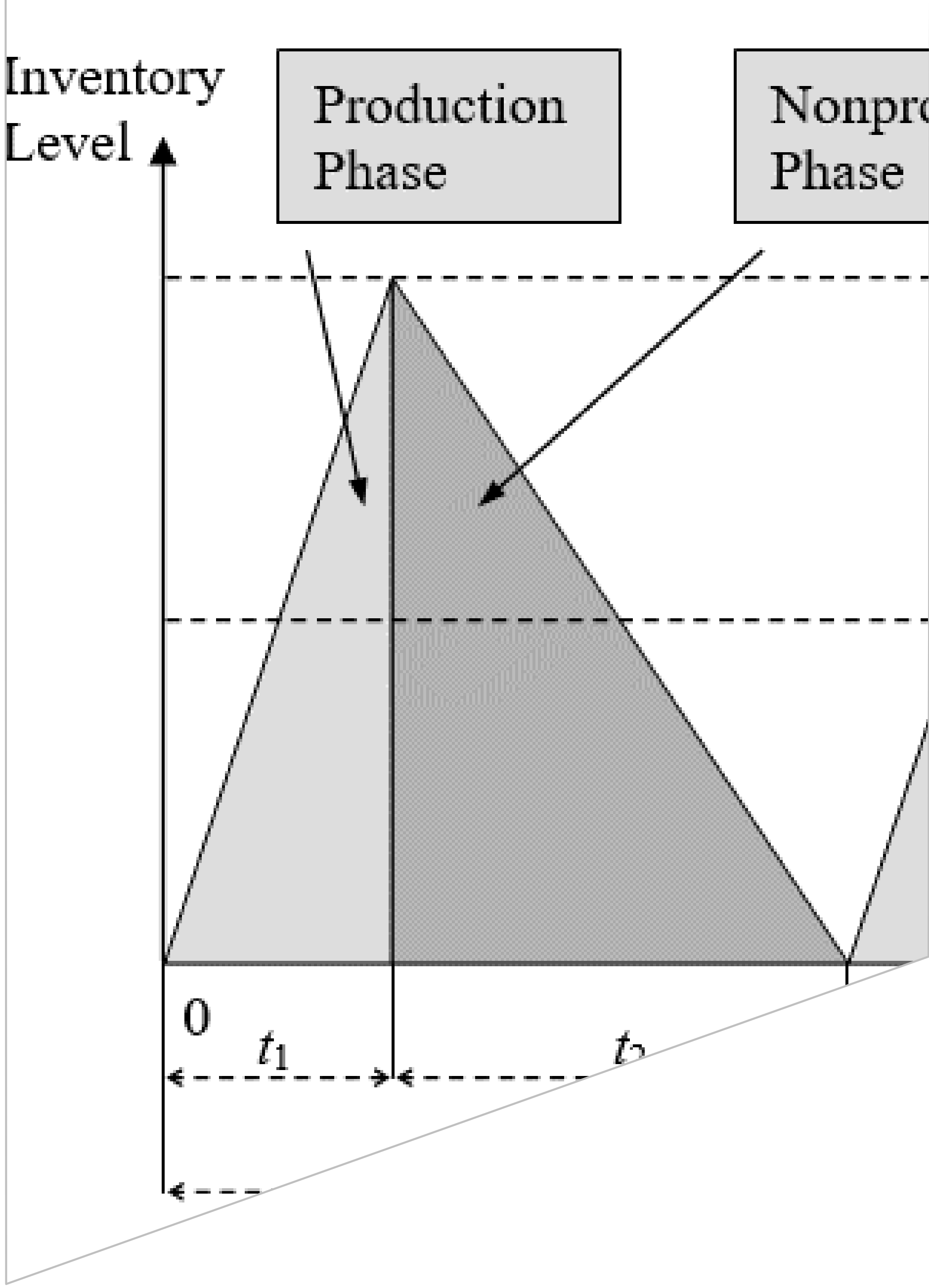


Fig. 31 – Probability analysis

5

Inventory Models



INTERNAL

Inventory Models

Introduction

- **Inventory**
  - Stored for use in future (fast and flexible availability, cost minimization).
  - Examples of inventories:
    - raw material,
    - finished goods,
    - semi-finished products,
    - spare parts.
- **Inventory management**
  - How much to order?
  - When to order?
  - Objective – minimization of total cost.

Inventory Models

Introduction

- **Partial inventory cost**
  - Holding & carrying cost
    - storage cost (place),
    - store keeping operations (movement),
    - insurance & taxes,
    - interest (investment),
    - spoilage & obsolescence.
  - Ordering cost
    - transport,
    - commission,
    - customs charge,
    - insurance.

Inventory Models

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Introduction

- **Inventory level**
  - Available size of the inventory (a number of stocked items, amount of stocked material, etc.).
- **Demand**
  - Rate of demand – amount of items or material required within a period.
- **Depletion**
  - Depletion rate – amount of stocked items or material moved from the warehouse, it is derived from the rate of demand.
  - Decreasing inventory level.
- **Replenishment**
  - Movement of delivered items or material into the warehouse.
  - Increasing inventory level.

Inventory Models

Introduction

- **Reordering**
  - Lead time – time interval between placing the order and delivery (receiving shipment).
  - Reorder point – inventory level at reordering.

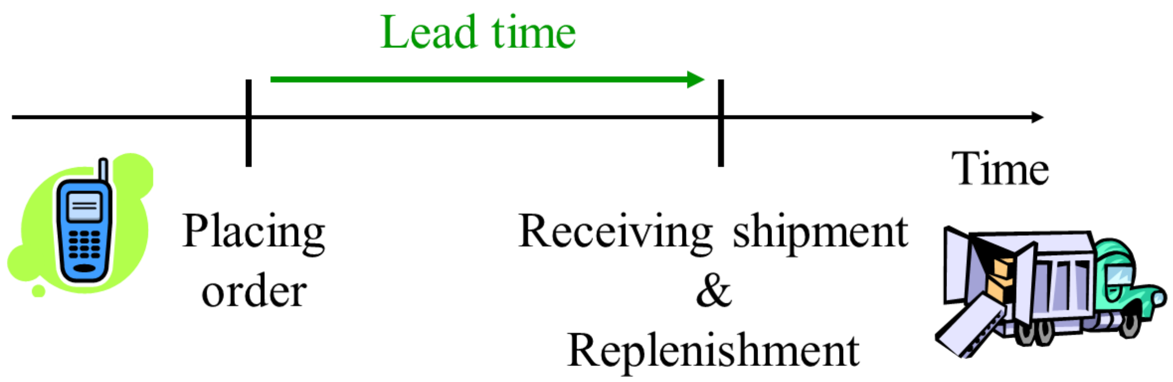

Fig. 32 – Reordering process

Inventory Models

Introduction

- **Shortage (stockout)**
  - Empty warehouse leads to unsatisfied demand (if there is no requirement during stockout period this is not registered as the shortage).

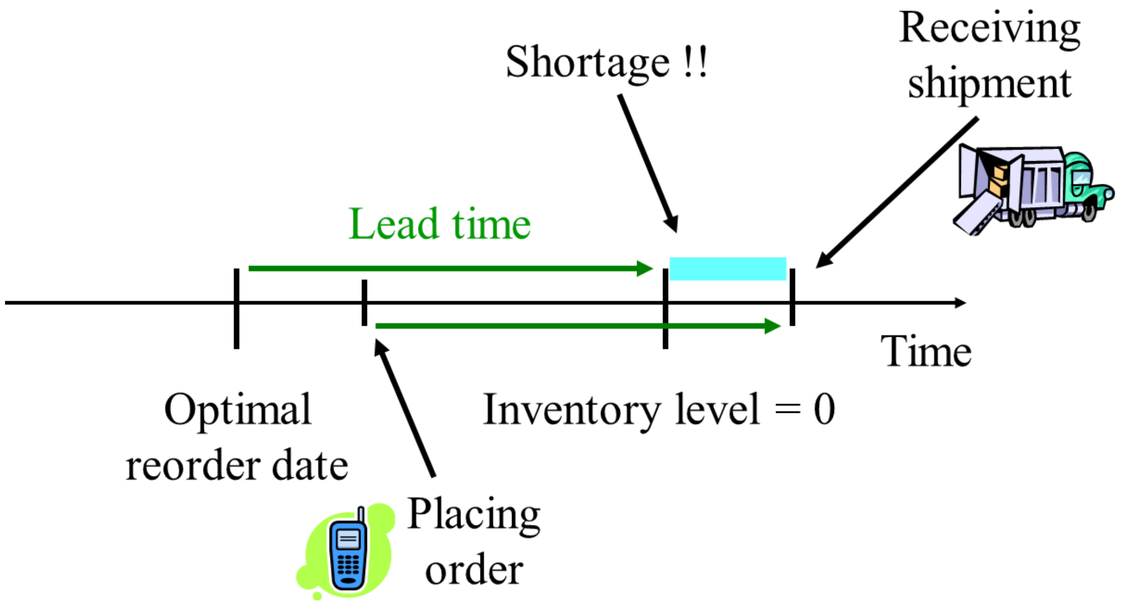

Fig. 33 – Shortage

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Inventory Models

Introduction

- **Safety stock**
  - In case of probabilistic demand.
  - Buffer is being built to prevent shortage. Generally, it is not possible to completely eliminate shortage (it depends on the type of probability distribution of demand).
- **Deterministic models**
  - All parameters are known with certainty (especially rate of demand and lead time).
- **Probabilistic models**
  - Some parameters are the values of random variables.

Inventory Models

Introduction

- **Demand classification**

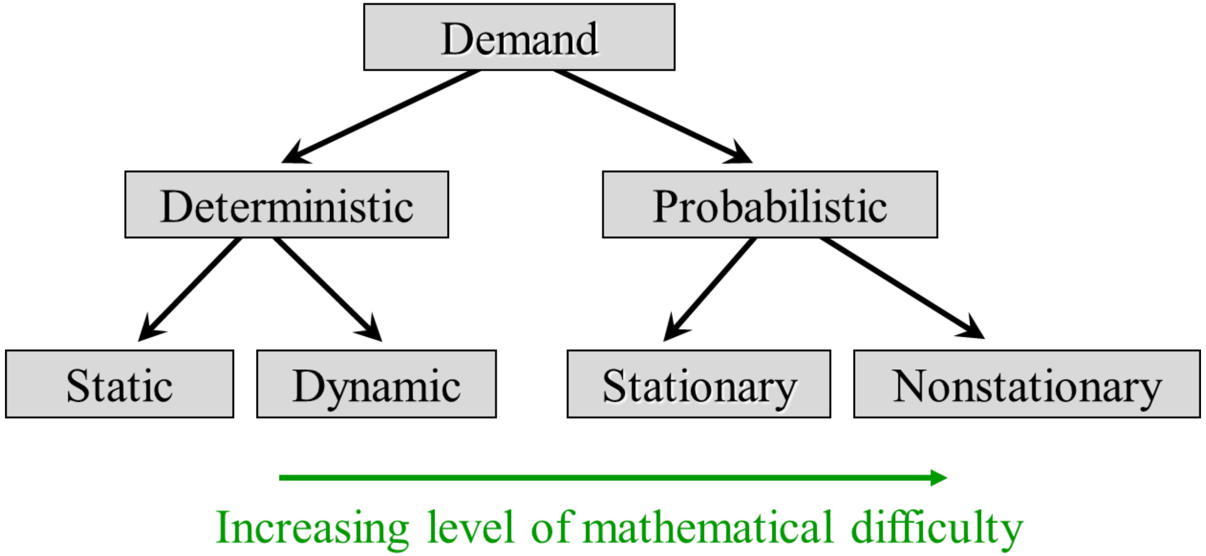

Fig. 34 – Demand classification

- **Static demand**
  - Rate of demand is known with certainty and it is constant in time.
- **Dynamic demand**
  - Rate of demand is known with certainty and it is not constant in time.
- **Stationary demand**
  - Probability distribution is unchanged over time.
- **Nonstationary demand**
  - Probability distribution varies in time.

Inventory Models

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Economic Order Quantity Model

- **Inventory management**
  - How much to order?
  - When to order?
  - What is the total cost?
  - What is the maximum inventory level?
  - What is the optimum length of the inventory cycle?

Inventory Models

Economic Order Quantity Model

- **Assumptions**
  - single item,
  - deterministic demand (static),
  - deterministic lead time (constant),
  - uniform depletion of the inventory,
  - constant order quantity,
  - purchasing cost is independent of the order quantity (no quantity discounts),
  - replenishment at one time,
  - no shortages, no surpluses (delivery exactly at the time of complete depletion).

Inventory Models

Economic Order Quantity Model

- **Inventory cycles**

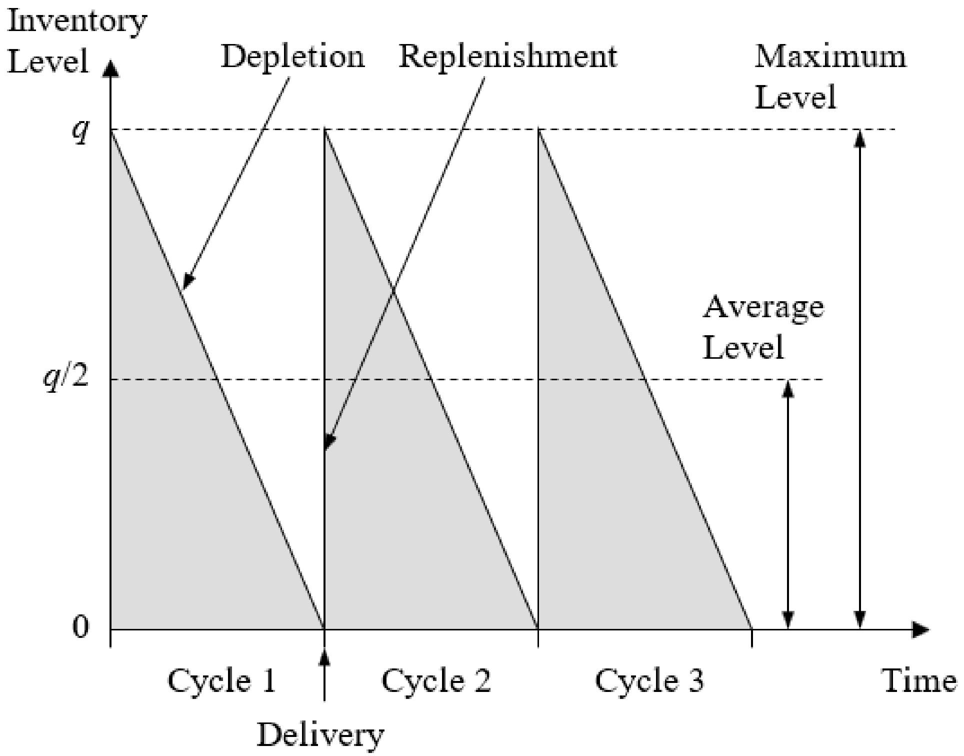

Fig. 35 – Inventory cycles in EOQ Model

Inventory Models

Economic Order Quantity Model

- **Example**
  - The private brewery produces monthly 4 000 hl of beer.
  - 25% of the production is planned to be filled into glass bottles. The empty bottles are stored in plastic cases (each case contains 20 bottles) and the average annual holding cost per case is 20 CZK.
  - The carrier, transporting the cases into the brewery's store, charges a fixed cost associated with each order for 11 000 CZK. In addition, the brewery's own fixed cost of 1 000 CZK per each order is necessary to be involved into the final calculation.
  - The lead time between the placing each order and its delivery is 1/2 of month. Since filling of the bottles is the uniform process the inventory depletion rate is uniform as well.
  - Management of the brewery decided to analyze the inventory system in order to minimize the total cost associated with inventory replenishment and holding of the bottles in the store.

Inventory Models

Economic Order Quantity Model

- **Input parameters and variables**
  - Annual demand $Q = 120\ 000$ cases
  - Unit annual holding cost $c_1 = 20$ CZK/case
  - Ordering cost $c_2 = 12\ 000$ CZK/order
  - Lead time $d = 1/2$ of month $= 1/24$ of year
  - Order quantity q
  - Number of orders within a year n
  - Length of inventory cycle t
  - Reorder point r

Inventory Models

Economic Order Quantity Model

- **Total annual cost**

$$TC = HC + OC$$

$$q_{\max} = q$$

$$q_{avg} = \frac{q}{2}$$

$$HC = c_1 q_{avg} = c_1 \frac{q}{2}$$

$$n = \frac{Q}{q}$$

$$OC = c_2 n = c_2 \frac{Q}{q}$$

$$\boxed{TC(q) = c_1 \frac{q}{2} + c_2 \frac{Q}{q}}$$

TC – total annual cost
HC – total annual holding cost
OC – total annual ordering cost
$q_{\max}$ – maximum inventory level
q_{avg} – average inventory level

Inventory Models

Economic Order Quantity Model

- **Total annual cost**

Tab. 28 – Cost calculation for 3 inventory policies

	Policy I	Policy II	Policy III
Annual demand Q	120 000	120 000	120 000
Order quantity q	10 000	60 000	120 000
Annual holding cost per case c_1	20	20	20
Average inventory level $q_{avg} = q/2$	5 000	30 000	60 000
Total annual holding cost *HC*	**100 000**	**600 000**	**1 200 000**
Ordering cost per order c_2	12 000	12 000	12 000
Number of orders $n = Q/q$	12	2	1
Total annual ordering cost *OC*	**144 000**	**24 000**	**12 000**
Total annual cost *TC*	**244 000**	**624 000**	**1 212 000**

$$TC(q) = c_1 \frac{q}{2} + c_2 \frac{Q}{q}$$

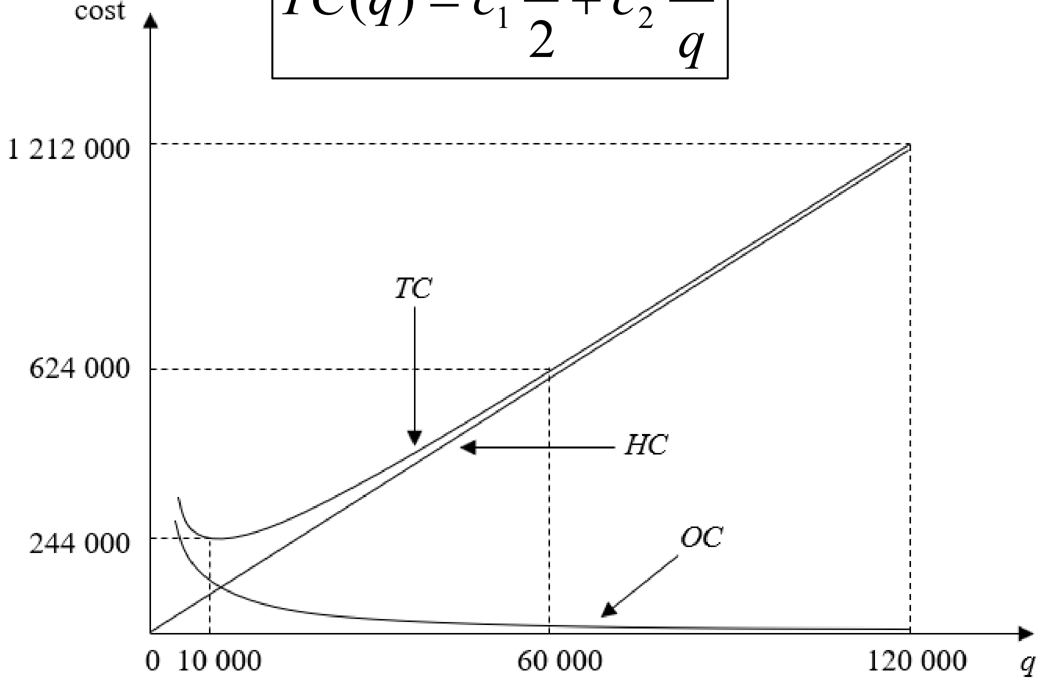

Fig. 36 – Cost charts for 3 inventory policies

Inventory Models

Economic Order Quantity Model

- **Inventory cycles**

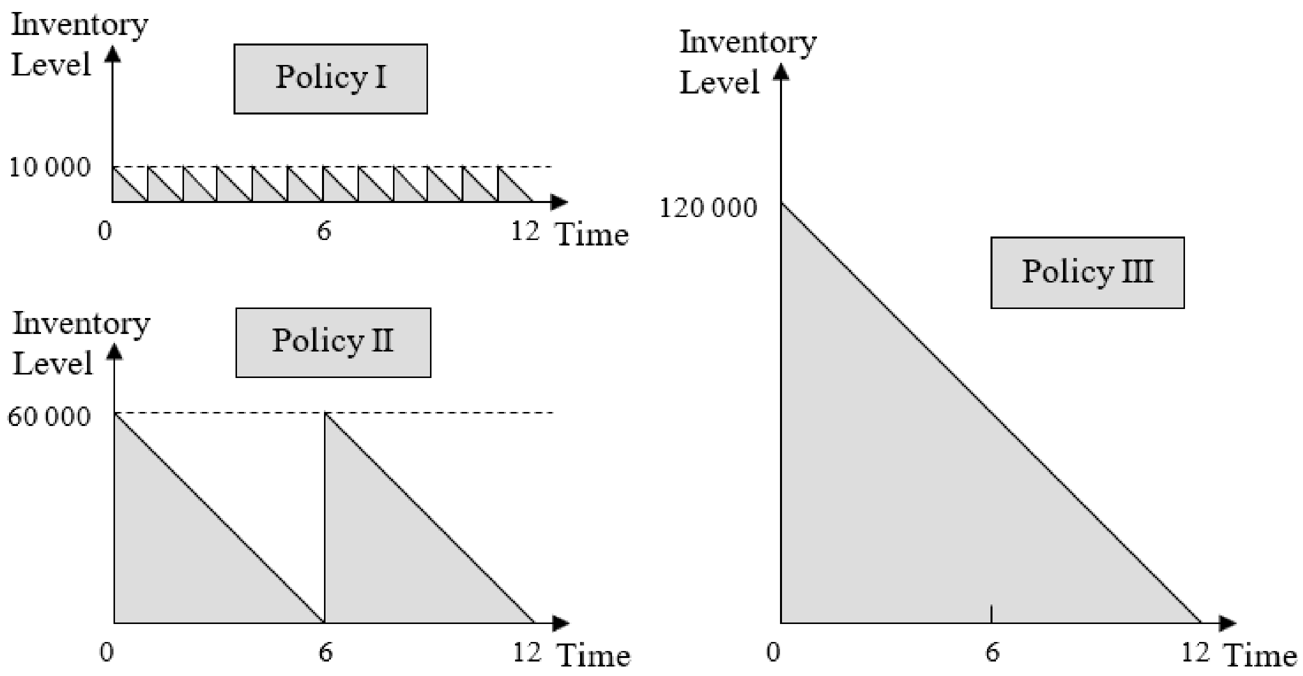

Fig. 37 – Inventory cycles for 3 inventory policies

Inventory Models

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Economic Order Quantity Model

- **Optimum order quantity**

$$TC(q) = c_1 \frac{q}{2} + c_2 \frac{Q}{q} \quad \rightarrow \min$$

$$\frac{dTC(q)}{dq} = \frac{c_1}{2} - \frac{c_2 Q}{q^2} = 0$$

$$\boxed{q^* = \sqrt{\frac{2Qc_2}{c_1}}} \qquad q^* = \sqrt{\frac{2(120\,000)(12\,000)}{20}} = 12\,000 \text{ cases}$$

- **Optimum total annual cost**

$$\boxed{TC^* = \sqrt{2Qc_1c_2}} \qquad TC^* = \sqrt{2(120\,000)(20)(12\,000)} = 240\,000 \text{ CZK}$$

Inventory Models

Economic Order Quantity Model

- **Optimum order quantity, optimum total annual cost**

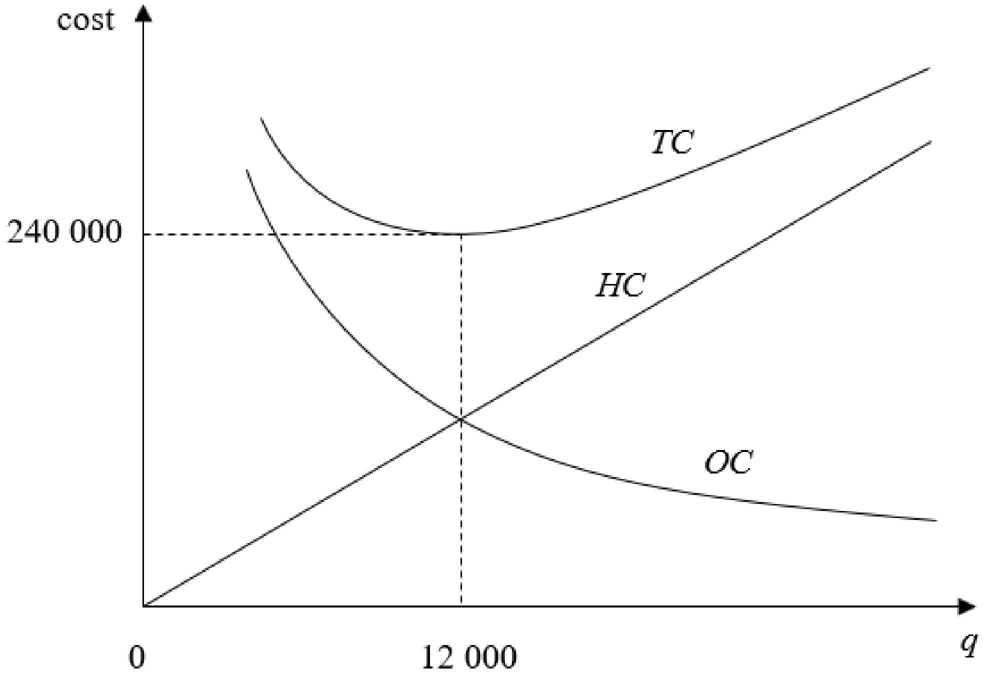

Fig. 38 – Optimum policy

Inventory Models

Economic Order Quantity Model

- **Optimum length of the inventory cycle**

$$t^* = \frac{1}{n^*} = \frac{q^*}{Q} = \sqrt{\frac{2c_2}{Qc_1}}$$

$t^* = 1/10$ of year

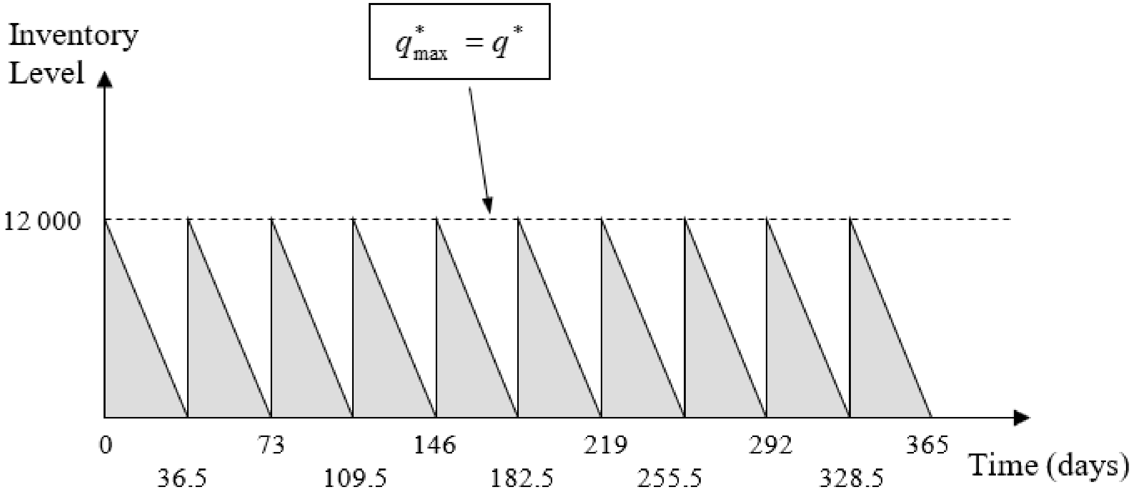

Fig. 39 – Optimum inventory cycles, maximum inventory level

Inventory Models

Economic Order Quantity Model

- **Optimum reorder point**

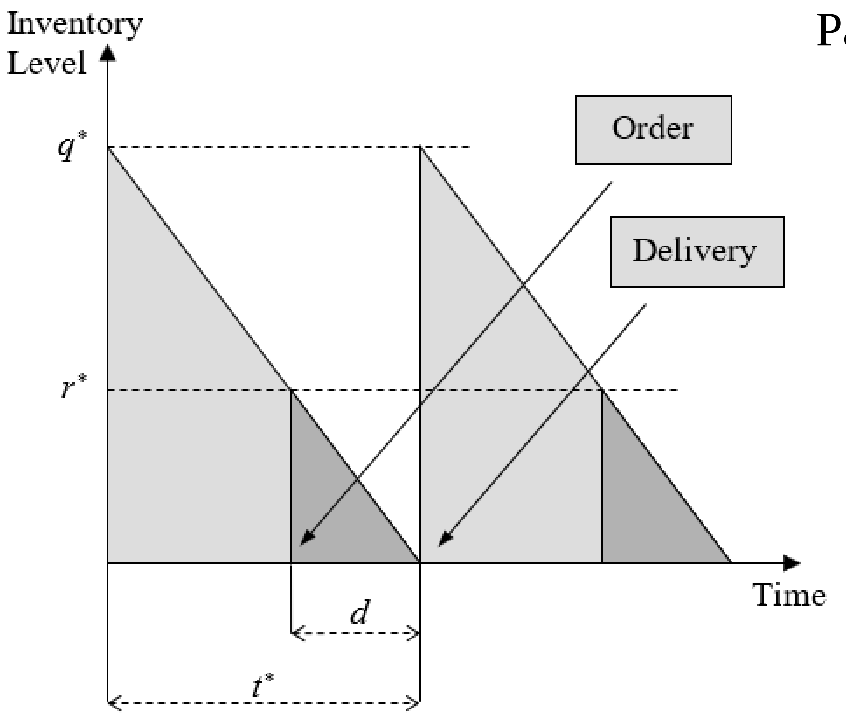

Fig. 40 – Optimum reorder point

Parallel triangles theorem: $$\frac{r^*}{d} = \frac{q^*}{t^*}$$

$$r^* = \frac{dq^*}{t^*} = dQ$$

$$r^* = \frac{1}{24} 120\,000 = 5\,000 \ \text{cases}$$

$$r^* = \frac{dq^*}{t^*} \mod q^* = dQ \mod q^*$$

Inventory Models

Economic Order Quantity Model

- **Example**
  - Firm dealing with the distribution of coal realizes the strategy of regular weekly orders (suppose 50 weeks in year).
  - Annual holding cost is 20 CZK per ton of coal, ordering cost is 800 CZK per order.
  - A switch to regular two-weeks ordering strategy did not cause any change in total annual cost.
  - Calculate the order quantity and total annual cost for previous and current strategy.
  - Find the optimum strategy.

Inventory Models

Economic Order Quantity Model

- **Input parameters and variables**
  - Unit annual holding cost $c_1 = 20$ CZK/ton
  - Ordering cost $c_2 = 800$ CZK/order
  - Number of orders for strategy 1 $n_1 = 50$/year
  - Number of orders for strategy 2 $n_2 = 25$/year
  - Order quantity in strategy 1 q
  - Order quantity in strategy 2 $2q$
  - Total annual cost in strategy 1 TC_1
  - Total annual cost in strategy 2 TC_2

Inventory Models

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Economic Order Quantity Model

- **Order quantity and total annual cost for both strategies**

$$TC_1 = TC_2$$

$$c_1 \frac{q}{2} + n_1 c_2 = c_1 \frac{2q}{2} + n_2 c_2$$

$$20 \frac{q}{2} + 50.800 = 20 \frac{2q}{2} + 25.800$$

$$q = 2000 \text{ tons}$$

$$2q = 4000 \text{ tons}$$

$$TC_1 = TC_2 = 60000 \text{ CZK}$$

$$Q = 50.2000 = 100000 \text{ tons}$$

- **Optimum order quantity and optimum total annual cost**

$$q^* = \sqrt{\frac{2.100000.800}{20}} \doteq 2828 \text{ tons}$$

$$TC^* = \sqrt{2.100000.20.800} \doteq 56570 \text{ CZK}$$

Inventory Models

Economic Production Lot Size Model (Production Order Quantity Model)

- **Inventory management**
  - What is the optimum lot size?
  - What is the maximum level of the inventory?
  - What is the total cost?
  - How long does the production process take?
  - When to start the preparation process (setup) for the production?

Inventory Models

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Economic Production Lot Size Model (Production Order Quantity Model)

- **Assumptions**
  - single item,
  - deterministic demand (static),
  - deterministic lead time (constant),
  - uniform depletion of the inventory,
  - constant lot size,
  - replenishment within production phase,
  - no shortages, no surpluses (production phase starts exactly at the time of complete depletion).

Inventory Models

Economic Production Lot Size Model (Production Order Quantity Model)

- **Inventory cycles**

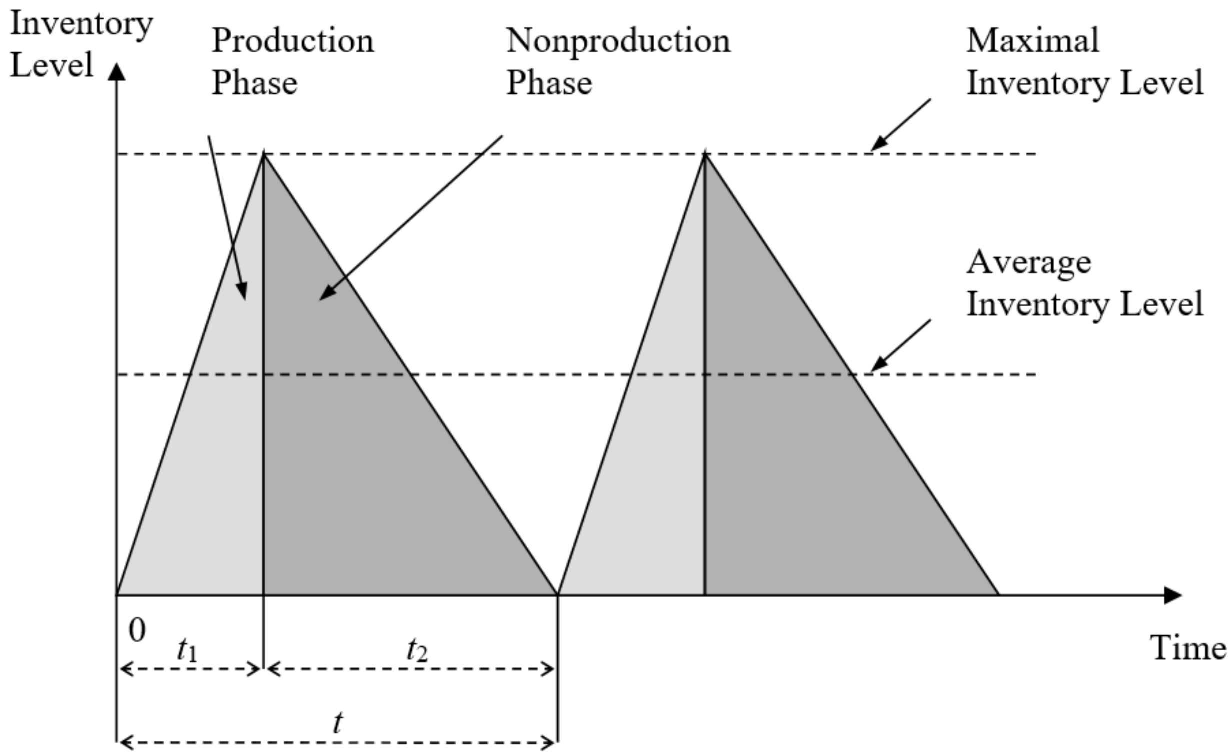

Fig. 41 – Inventory cycles in POQ Model

Inventory Models

Economic Production Lot Size Model (Production Order Quantity Model)

- **Inventory cycles**
  - Production phase
    - production (production rate),
    - demand (demand rate),
    - replenishment.
  - Nonproduction (depletion) phase
    - demand (demand rate),
    - depletion.

production rate > demand rate

Inventory Models

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Economic Production Lot Size Model (Production Order Quantity Model)

- **Example**
  - The private brewery produces monthly 4 000 hl of beer.
  - 25% of the production is planned to be filled into glass bottles. The empty bottles are stored in plastic cases (each case contains 20 bottles) and the average annual holding cost per case is 20 CZK.
  - Empty bottles are processed on the cleaning line, daily output is 8 000 bottles.
  - Setup cost was calculated to 12 000 CZK per one successive cleaning process.
  - Preparation of the cleaning line takes 1/2 of month (lead time).
  - The brewery's management wants to determine the size of cleaning batch to minimize the total annual cost.

Inventory Models

Economic Production Lot Size Model (Production Order Quantity Model)

- **Example**
  - The private brewery produces monthly 4 000 hl of beer.
  - 25% of the production is planned to be filled into glass bottles. The empty bottles are stored in plastic cases (each case contains 20 bottles) and the average annual holding cost per case is 20 CZK.
  - Empty bottles are processed on the cleaning line, daily output is 8 000 bottles.
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  - Preparation of the cleaning line takes 1/2 of month (lead time).
  - The brewery's management wants to determine the size of cleaning batch to minimize the total annual cost.

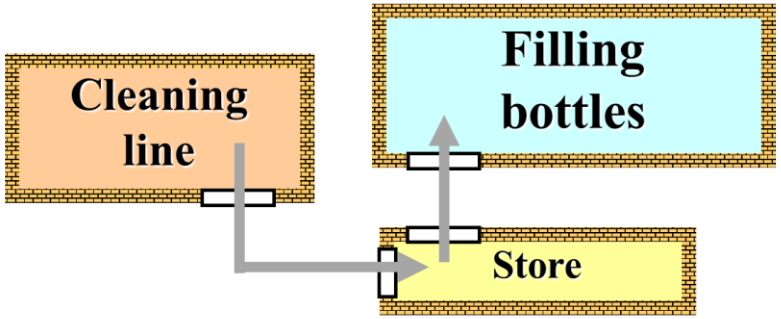

Fig. 42 – Production phase

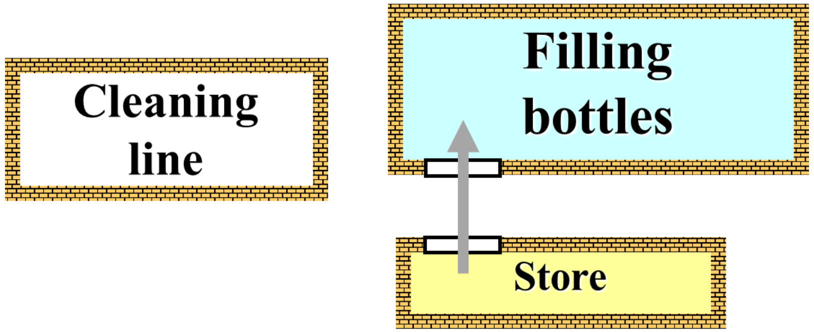

Fig. 43 – Nonproduction phase

Inventory Models

Economic Production Lot Size Model (Production Order Quantity Model)

- **Input parameters and variables**
  - Annual demand — $Q = 120\ 000$ cases
  - Unit annual holding cost — $c_1 = 20$ CZK/case
  - Fixed setup cost per cleaning lot — $c_2 = 12\ 000$ CZK/lot
  - Lead time — $d = 1/2$ of month $= 1/24$ of year
  - Production rate — $p = 146\ 000$ cases per year
  - Demand rate — $h = 120\ 000$ cases per year
  - Production lot size — q
  - Number of lots within a year — n
  - Length of inventory cycle — t
  - Length of production period — t_1
  - Length of depletion period — t_2
  - Starting setup point — r

Inventory Models

Economic Production Lot Size Model (Production Order Quantity Model)

- **Total annual cost**

$$TC = HC + SC$$

$$q = pt_1$$

$$q_{\max} = pt_1 - ht_1 = (p-h)t_1 = \frac{p-h}{p} q$$

$$q_{avg} = \frac{q_{\max}}{2} = \frac{p-h}{p} \frac{q}{2}$$

$$HC = c_1 q_{avg} = c_1 \frac{p-h}{p} \frac{q}{2}$$

$$n = \frac{Q}{q}$$

$$SC = c_2 n = c_2 \frac{Q}{q}$$

$$\boxed{TC(q) = c_1 \frac{p-h}{p} \frac{q}{2} + c_2 \frac{Q}{q}}$$

TC – total annual cost
HC – total annual holding cost
SC – total annual setup cost
$q_{\max}$ – maximum inventory level
q_{avg} – average inventory level

Inventory Models

Economic Production Lot Size Model (Production Order Quantity Model)

- **Optimum lot size**

$$TC(q) = c_1 \frac{p-h}{p}\frac{q}{2} + c_2 \frac{Q}{q} \quad \to \min$$

$$\frac{dTC(q)}{dq} = \frac{c_1}{2}\frac{p-h}{p} - \frac{c_2 Q}{q^2} = 0$$

$$\boxed{q^* = \sqrt{\frac{2Qc_2}{c_1}}\sqrt{\frac{p}{p-h}}}$$

$$q^* = \sqrt{\frac{2(120\,000)(12\,000)}{20}}\sqrt{\frac{146\,000}{146\,000 - 120\,000}} \doteq 28\,436.16 \text{ cases.}$$

Inventory Models

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Economic Production Lot Size Model (Production Order Quantity Model)

- **Optimum total annual cost**

$$TC^* = \sqrt{2Qc_1c_2}\sqrt{\frac{p-h}{p}}$$

$$TC^* = \sqrt{2(120\,000)(20)(12\,000)}\sqrt{\frac{146\,000-120\,000}{146\,000}} \doteq 101\,279.49 \text{ CZK.}$$

Inventory Models

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Economic Production Lot Size Model (Production Order Quantity Model)

- **Optimum length of the production period**

$$t_1^* = \frac{q^*}{p} \qquad t_1^* = 0.1948 \text{ years} = 71.1 \text{ days}$$

- **Optimum length of the depletion period**

$$t_2^* = \frac{q_{\max}^*}{h} = \frac{p-h}{ph} q^* \qquad t_2^* \doteq 0.0422 \text{ years} = 15.4 \text{ days}$$

- **Optimum length of the inventory cycle**

$$t^* = t_1^* + t_2^* \qquad t^* \doteq 0.1948 + 0.0422 = 0.237 \text{ years} = 86.5 \text{ days}$$

Inventory Models

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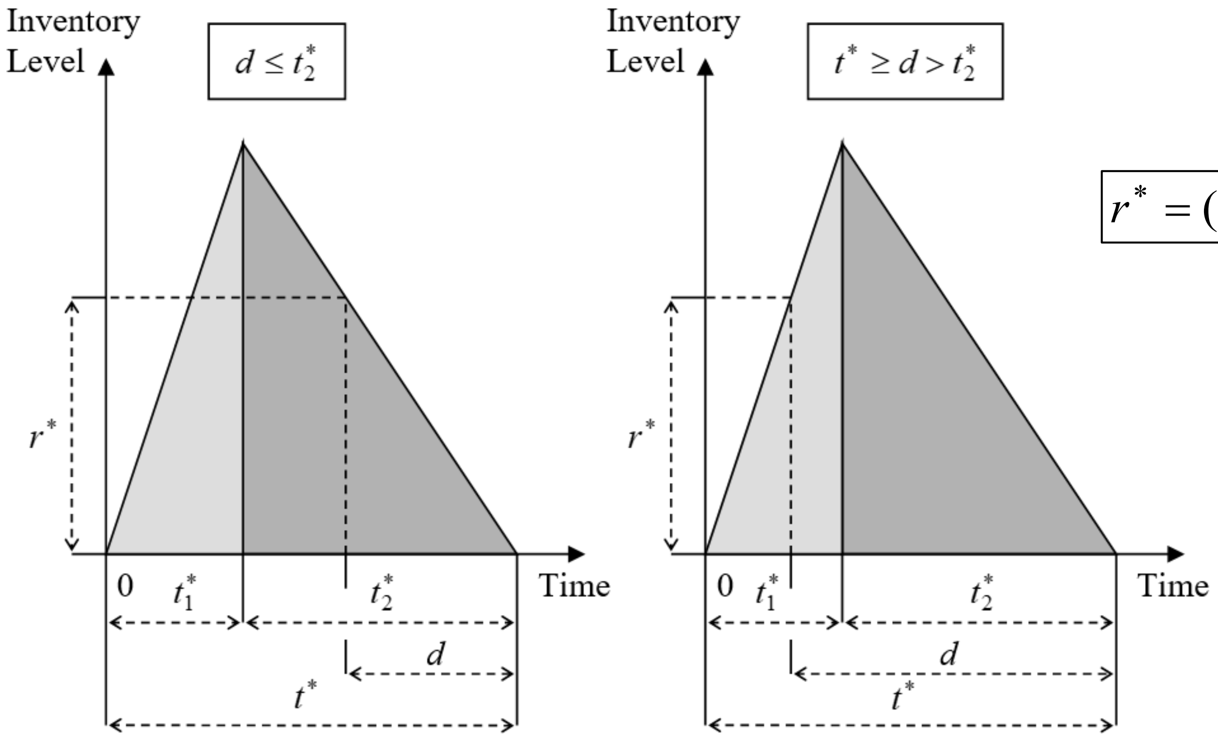
Economic Production Lot Size Model (Production Order Quantity Model)

- **Optimum starting setup point**

$$r^* = hd$$



$$r^* = (p-h)\,(t^* - d)$$

Fig. 44 – Optimum starting setup point

Inventory Models

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Economic Production Lot Size Model (Production Order Quantity Model)

- **Optimum starting setup point**

$$d = 1/24 \doteq 0.0417 < t_2^* \doteq 0.0422$$

$$r^* = hd = 5\,000 \text{ cases}$$

- **Optimum maximal inventory level**

$$\boxed{q_{\max}^* = ht_2^*} \qquad q_{\max}^* \doteq 5064 \text{ cases}$$

Inventory Models

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Single-Period Decision Model

- **Assumptions**
  - single item,
  - probabilistic demand,
  - only one order in time period (no reorder is possible),
  - two situations at the end of period:
    - surplus,
    - stockout.
- **Examples of seasonal and perishable items**
  - newspapers (newsboy problem), bread, flowers, fruits, seasonal clothing, Christmas trees.

Inventory Models

Single-Period Decision Model

- **Example**
  - A bakery department's manager of a new supermarket should optimize everyday order of rolls.
  - The supermarket buys a roll for 1 CZK and sells it to the final consumers for 2 CZK. If, in the evening some rolls remain in the store, the bakery department changes them into crumbs that will be sold later for 12 CZK per sack. For filling one sack of crumbs 20 rolls are needed.
  - From the experience of opening comparable supermarkets in comparable areas, the supermarket's analyst recommends to consider the normal probability distribution of the daily demand with a mean of 10 000 rolls and a standard deviation of 500 rolls.

Inventory Models

Single-Period Decision Model

- **Input parameters and variables**
  - mean value of daily demand — μ = 10 000 rolls
  - standard deviation of daily demand — σ = 500 rolls
  - purchase cost — 1 CZK/roll
  - selling price — 2 CZK/roll
  - salvage value — 12/20 = 0.6 CZK/roll
  - real daily demand for rolls — Q
  - daily quantity of ordered rolls — q
  - probability with which no stockout occurs (service level) — p

Inventory Models

Single-Period Decision Model

- **Marginal loss**
  - The real demand is less than the order quantity ($Q < q$)

$$ML = \text{purchase cost} - \text{salvage value} \qquad ML = 1 - 0.6 = 0.4 \text{ CZK/roll}$$

  - Expected marginal loss

$$p(ML)$$

- **Marginal profit loss**
  - The real demand is greater than the order quantity ($Q > q$)

$$MPL = \text{selling price} - \text{purchase cost} \qquad MPL = 2 - 1 = 1 \text{ CZK/roll}$$

  - Expected marginal profit loss

$$(1 - p)(MPL)$$

Inventory Models

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Single-Period Decision Model

- **Optimum service level**

$$p(ML) = (1 - p)(\text{MPL})$$

$$\boxed{p = \frac{MPL}{ML + MPL}} \qquad p = \frac{1}{0.4 + 1} \doteq 0.7143$$

- Optimum order quantity

$$\text{P}\{Q \le q\} \ge p$$

$$z_p = \frac{Q - \mu}{\sigma} \quad \rightarrow \quad Q = z_p\sigma + \mu$$

$$q \ge \mu + z_p\sigma$$

$$\boxed{q^* = \mu + z_p\sigma} \qquad q^* = 10\,000 + 0.566\,(500) = 10\,283 \text{ rolls}$$

Inventory Models

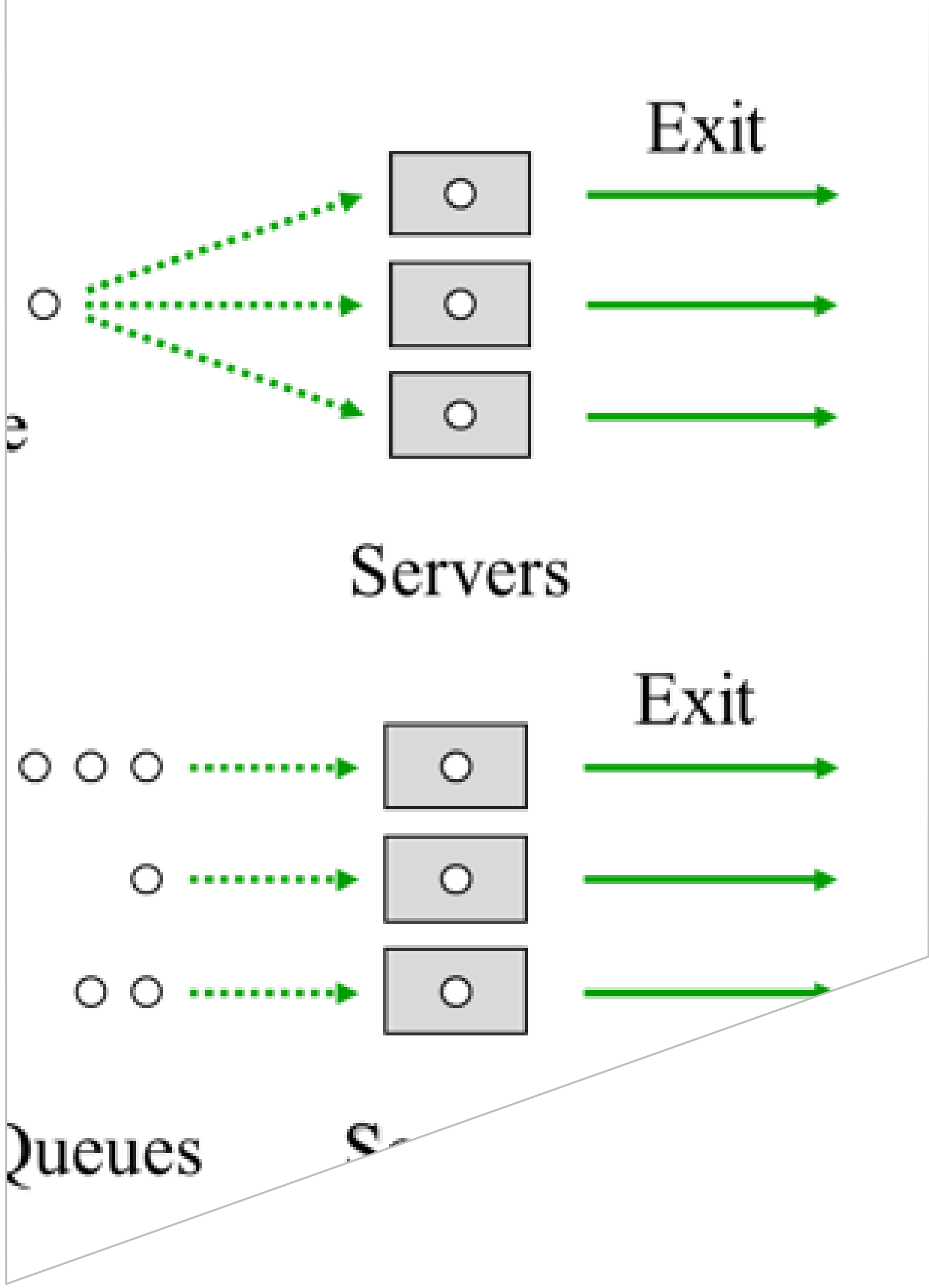
Single-Period Decision Model

- **Example**
  - In previous example we consider the uniform probability distribution of the daily demand in the interval <9 000, 11 000>.
- **Optimum order quantity**

$$q^* = 9\ 000 + 0{,}7143(11\ 000 - 9\ 000) \doteq 10\ 429 \text{ rolls}$$

6

Waiting Line Models



Waiting Line Models

Introduction

- **Waiting line system**
  - Two objects in the system
    - customer,
    - server (service facility).

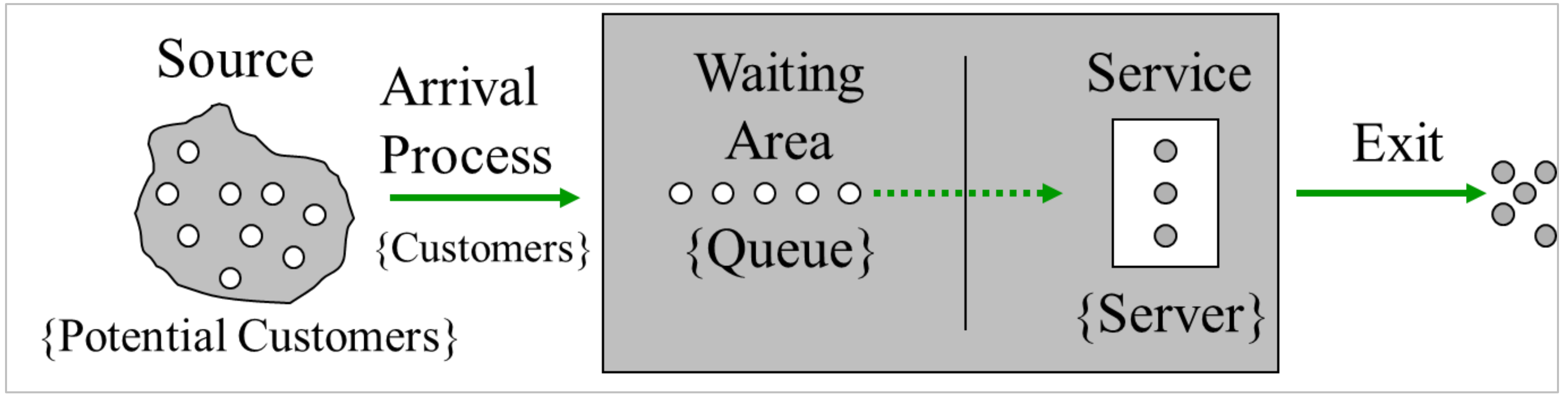


Fig. 45 – Scheme of waiting line system

Waiting Line Models

Introduction

- **Waiting line system**
  - Examples of real system

Tab. 29 – Real waiting line systems

Service System	Customer	Server
Doctor’s consultancy room	Patient	Doctor
Bank	Client	Clerk
Crossing	Car	Traffic lights
Telephone exchange	Call	Switchboard
Airport	Airplane	Runway
Fire station	Fire	Emergency unit
Service station	Car	Petrol pump

Waiting Line Models

Introduction

- **Source of customers**
  - Size of the source
    - infinite (tourists),
    - finite (machines in a factory).
- **Arrival process**
  - Way of arrivals
    - individual arrivals (patients),
    - arrivals in batches (group of tourists).
  - Time of arrivals
    - scheduled arrivals (trains),
    - unscheduled arrivals (patients).

Waiting Line Models

Introduction

- **Arrival process**
  - Number of customers
    - arrival rate – number of arrivals per time unit (Poisson probability distribution),
    - λ = average arrival rate – average number of arrivals per time unit.
  - Time of customers‘ arrivals
    - interarrival time – time period between two subsequent arrivals (exponential probability distribution),
    - $1/\lambda$ = average interarrival time – average time period between two subsequent arrivals.

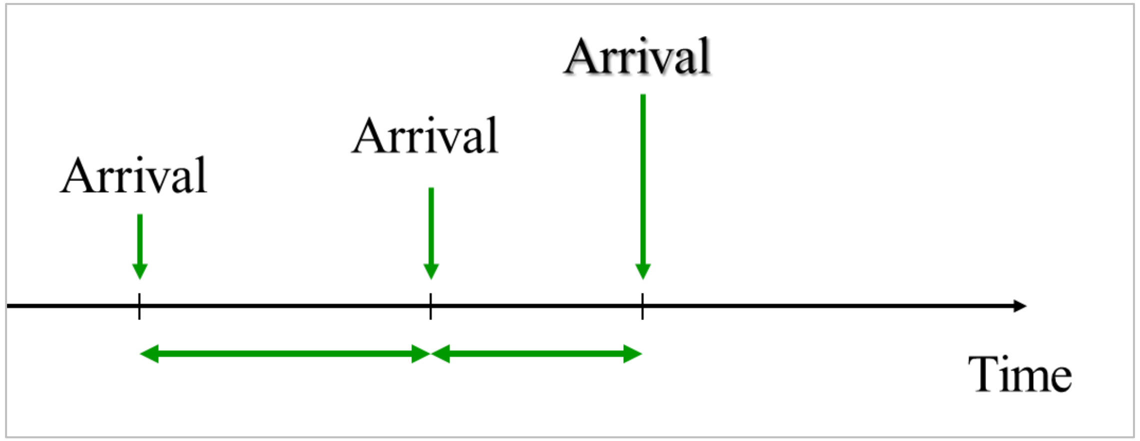


Fig. 46 – Arrivals of customers

INTERNAL

Waiting Line Models

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Introduction

- **Service process**
  - Number of customers
    - service rate – number of customers served per time unit (Poisson probability distribution),
    - μ = average service rate – average number of customers served per time unit.
  - Time of service
    - service time – time the customer spends at service facility (exponential probability distribution),
    - $1/\mu$ = average service time – average time customers spend at service facility.

Waiting Line Models

Introduction

- **Service configuration definition**
  - Type of facility – type of the offered service.
  - Number of facilities – one or multiple facilities.
  - Arrangement of service facilities – network of related or unrelated facilities.
- **Service configurations**
  1. Single facility

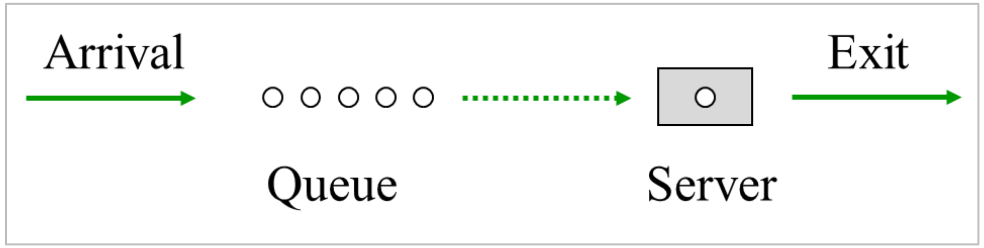

Fig. 47 – Single facility

Waiting Line Models

Introduction

- **Service configurations**
  2. Multiple, parallel facility

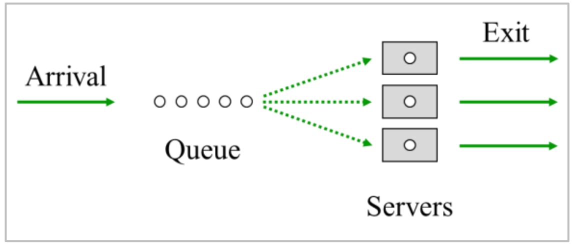

Fig. 48 – Identical facilities, single queue

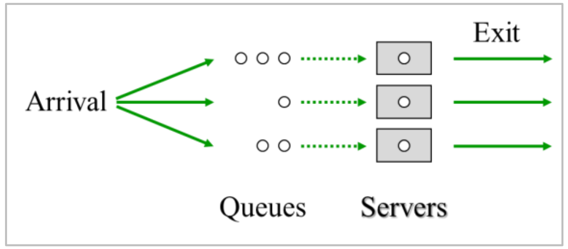

Fig. 49 – Identical facilities, multiple queues

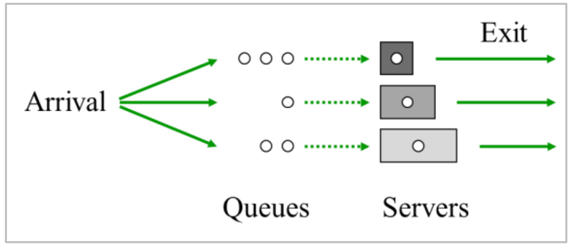

Fig. 50 – Nonidentical facilities

  3. Multiple, serial facilities

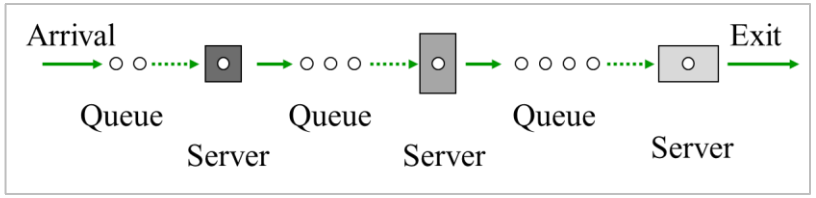

Fig. 51 – Serial facilities

  4. Combination of parallel and serial arrangement of facilities

Waiting Line Models

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Introduction

- **Discipline of the queue**
  - FCFS (First-Come, First-Served) – FIFO (First-In, First-Out)
  - LCFS (Last-Come, First-Served) – LIFO (Last-In, First-Out)
  - PRI (priority system)
  - SIRO (Selection In Random Order)
- **Classification of waiting line models (Kendall‘s notation)**

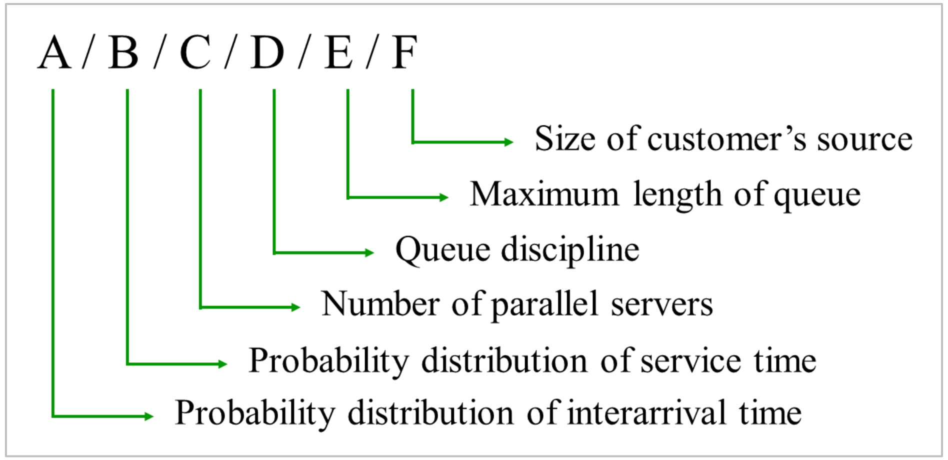

Fig. 52 – Kendall‘s notation of parallel facilities models

Waiting Line Models

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Standard Single-Server Exponential Model

- **Assumptions**
  - Kendall's notation: M/M/1/FCFS/∞/∞,
  - single facility,
  - exponential probability distribution of interarrival time (λ = average arrival rate),
  - exponential probability distribution of service time (μ = average service rate)
  - queue discipline is FCFS,
  - unlimited length of the queue,
  - infinite size of the source of customers,
  - $\mu > \lambda$ stability of the system.

Waiting Line Models

ŠKODA AUTO University

Standard Single-Server Exponential Model

- **Example**
  - In the small grocery store, there is only one shop board with only one shop assistant.
  - In the period from 8 a.m. to 6 p.m. 18 customers per hour (on the average) come to the grocery.
  - The assistant is able to serve (on the average) 25 customers per hour.
  - Analyze the behavior of the waiting line system.
- **Input parameters**
  - Average arrival rate $\lambda = 18$ customers per hour
  - Average service rate $\mu = 25$ customers per hour

Waiting Line Models

ŠKODA AUTO University

Standard Single-Server Exponential Model

- **Measures of performance**
  - Utilization of the system
    - probability that the server is busy,
    - probability that there is at least one customer in the system,
    - probability that the arriving customer must wait in the queue.

$$\rho = \frac{\lambda}{\mu} \qquad \rho = 0.72$$

  - Probability of an empty facility
    - probability that the server is idle,
    - probability that the arriving customer will not wait in the queue.

$$P(0) = 1 - \rho \qquad P(0) = 0.28$$

Waiting Line Models

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Standard Single-Server Exponential Model

- **Measures of performance**
  - Probability of finding exactly N customers in the system

$$P(N) = P(0)\rho^N = (1-\rho)\rho^N$$

Tab. 30 – Probabilities of finding *N* customers in grocery

$P(0)$	0.280
$P(1)$	0.202
$P(2)$	0.145
$P(3)$	0.105
$P(4)$	0.075
$P(5)$	0.054

$$P(0) + P(1) + P(2) + P(3) + P(4) + \ldots = 1$$

Waiting Line Models

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Standard Single-Server Exponential Model

- **Measures of performance**
  - Average waiting time in the system

$$W = \frac{1}{\mu - \lambda}$$

$$W \doteq 0.143 \text{ hours} \doteq 8.6 \text{ min}$$

  - Average waiting time in the queue

$$W_q = W - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$W_q \doteq 0.103 \text{ hours} \doteq 6.2 \text{ min}$$

Waiting Line Models

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Standard Single-Server Exponential Model

- **Measures of performance**
  - Average number of customers in the system

$$L = \lambda W = \frac{\lambda}{\mu - \lambda}$$

$L \doteq 2.57$ customers

  - Average number of customers in the queue

$$L_q = \lambda W_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$L_q \doteq 1.85$ customers

Waiting Line Models

Standard Multi-Server Exponential Model

- **Assumptions**
  - Kendall's notation: M/M/K/FCFS/∞/∞, ,
  - K multiple, parallel facilities, single queue
  - exponential probability distribution of interarrival time (λ = average arrival rate),
  - exponential probability distribution of service time (μ = average service rate)
  - queue discipline is FCFS,
  - unlimited length of the queue,
  - infinite size of the source of customers,
  - $K\mu>\lambda$ stability of the system.

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Thank you for attention

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